ALGORITHMS FOR CERTAIN COMPUTATIONAL MATHEMATICS PROBLEMS

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ABSTRACT

Algorithms for Certain Computational Mathematics Problems

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This dissertation presents algorithms and explores diverse aspects of number theory, cryptographic systems, and partition theory. In Chapter Two, attention is focused on enhancing the security of the extended Rabin cryptosystem by incorporating multiple prime numbers into the encryption process, thereby increasing the complexity of decryption and fortifying resilience against quantum computing threats. Additionally, experimental results corroborate the efficacy of proposed algorithms, aligning closely with existing decryption methods while offering enhanced versatility.

Chapter Three presents a detailed exploration of sums of powers of arithmetic progressions, offering simplified formulas and algorithms for efficient computation, leveraging Stirling and Eulerian numbers. A comparison with existing methods underscores the computational efficiency of the proposed approaches.

In Chapter Four, properties and algorithms related to Ramanujan-type cubic equations are elucidated, showcasing a comprehensive computational methodology and its application through examples and cubic Shevelev sums.

Chapter Five extends the understanding of Leonardo sequences and second-order non-homogeneous recursive sequences, unveiling novel identities and combinatorial results. These findings are applied to investigate series representations, enriching the discourse on number theory.

Lastly, Chapter Six investigates the representation of positive odd integers as the sum of arithmetic progressions, building upon historical and contemporary works to provide theorems and efficient algorithms for computing such representations. This dissertation contributes to diverse

areas within mathematics, cryptography, and computational methods, promising new avenues for exploration and application.

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DEDICATION

I dedicate this dissertation to my parents.

媽媽，謝謝妳把我撫養長大，讓我有機會拿到這個博士學位。謝謝妳一路以來的支持與鼓勵。妳為了我，跟著爸爸來到美國，一個妳需要重新開始的地方，妳是我的英雄。這個博士畢業論文獻給妳。

-永遠愛妳的兒子。

Thank you Dad for bringing Mom and me to the USA. I couldn't have done this without you. I know our time together wasn't as long as either of us liked, but I enjoyed every second of you being the only father figure I ever had. I dedicate this dissertation to you. I love you, dad.

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CHAPTER 1

INTRODUCTION

A computer would deserve to be called intelligent if it could deceive a human into believing that it was human.

-Alan Turing

This dissertation presents algorithms and explores diverse aspects of combinatorial number theory, cryptographic systems, and partition theory. It begins by introducing several theorems and algorithms concerning the Chinese Remainder Theorem and the extended Rabin cryptosystem. In subsequent chapters, the dissertation delves into sums of powers of arithmetic progressions, Ramanujan-type cubic equations, Leonardo sequences, and the representation of positive integers as the sum of arithmetic progressions, offering novel insights, methodologies, and practical applications. These comprehensive explorations contribute to various fields within mathematics and cryptography, promising new avenues for research and application.

In Chapter Two, we introduced several theorems and algorithms concerning the Chinese Remainder Theorem and the extended Rabin cryptosystem. This cryptosystem allows for a key comprised of any finite number of unique primes. Additionally, we suggested approaches to relax the constraint on these primes, albeit with compromises in time complexity. These proposed algorithms notably bolster security by incorporating more prime numbers into the public key, thus heightening the difficulty of decryption. We also explored leveraging these algorithms to fortify security against quantum computing threats. Furthermore, we conducted extensive experimentation across six sizable datasets. The results of these experiments demonstrate that our proposed algorithms closely align with the existing decryption algorithm in the Rabin cryptosystem, specifically when the key consists of two distinct primes, all while maintaining enhanced versatility. See Zhan et al. [99] for the publication.

In Chapter Three, we examine the sums of powers of arithmetic progressions, denoted as $a^p +$

$(a+d)^p + (a+2d)^p + \cdots + (a+(n-1)d)^p$, where $n \geq 1$, p is a non-negative integer, and a and d are complex numbers with $d \neq 0$. We provide a straightforward proof of a theorem initially presented by Laissaoui and Rahmani [47] as well as an algorithm derived from their formula. Furthermore, we introduce a simplified version of Laissaoui and Rahmani's formula, tailored for computational efficiency, along with a second algorithm based on this simplification. Both formulations utilize Stirling numbers of the second kind. Alternatively, the sums can be computed using Eulerian numbers, for which we propose a new method employing classical Eulerian numbers. The existing formula utilizing general Eulerian numbers tends to be more algorithmically intricate due to the increased number of computations required. To address this, we introduce two innovative algorithms that incorporate both types of Eulerian numbers. Finally, we compare our findings to the results obtained by Xiong et al. [97], who utilized general Eulerian numbers in their approach. Finally, we give a table for Faulhaber's formula in the form of $\sum_{k=1}^{n} k^p$, where $p \in \mathbb{N}$. See Shiue et al. [80] and Shiue et al. [79] for the publications.

In Chapter Four, we explore the characteristics of a theorem attributed to Ramanujan in order to establish properties and algorithms pertaining to cubic equations. Subsequently, we devise a comprehensive computational method for constructing Ramanujan-type equations from arbitrary general cubic equations, along with a cosine Ramanujan-type identity. This procedure was formulated based on a thorough examination of the properties inherent in Ramanujan-type cubic equations. To illustrate the application of these concepts, we include examples along with cubic Shevelev sums. See Shiue et al. [82] and Shiue et al. [83] for the publications. It was also presented at JMM Boston 2023 Huang et al. [35].

In the fifth chapter, we build upon the recent discoveries made by several researchers regarding the characteristics of elements in the Leonardo sequence, placing them in the context of second-order recursive sequences. We achieve this expansion by exploring the properties of difference equations in the homogeneous Fibonacci sequence and the non-homogeneous characteristics of the Leonardo sequences. Through this exploration, we uncover a range of new identities associated with a generalized Leonardo sequence and its algorithm. Moreover, this investigation reveals a series of combinatorial findings that highlight the sophisticated properties of hyper-Fibonacci numbers compared to traditional Fibonacci numbers, including their interaction with Leonardo numbers. Furthermore, we delve into various second and third order recursive sequences, meticulously formulating properties related to the roots of their characteristic equations. While some of these properties are well-established, many are newly discovered. These findings are then applied to novel investigations of series representations, including their convergence criteria, expressed as $\sum_{n=0}^{\infty} \frac{a_{mn}}{10^{n+1}}, \quad m = 1, 2, 3,$ and are extended to numerous standard sequences, treated as specific instances of a generic sequence $\{a_n\}$. The detailed exposition of the algebra underlying these theorems, along with their associated lemmas and corollaries, promises to unveil new avenues of exploration for interested number theorists. The concluding results regarding series values and approximations further enrich the discussion. See Shannon et al. [75] for the publication. It was also presented at JMM San Francisco 2024 Huang et al. [36]. The Series part has been submitted for publication.

In the sixth and final chapter, we investigate the mathematical interest in representing certain positive integers as the sum of arithmetic progressions. Since 1844, interest has persisted in representing numbers as the sum of consecutive integers, initially explored by Wheatstone [94], Sylvester and Franklin [85] established Sylvester's Theorem in 1882, prompting numerous extensions, including sums of various arithmetic progressions and powers. Recent works by Ho et al. [29] and Ho et al. [30] extended Sylvester's Theorem to compute the ways a positive integer can be represented as the sum of all possible arithmetic progressions. We present theorems and efficient algorithms to compute the number of ways certain positive integers can be written as the sum of arithmetic progressions, as well as to list such ways.

CHAPTER 2

TOWARDS A NOVEL GENERALIZED CHINESE REMAINDER ALGORITHM FOR EXTENDED RABIN CRYPTOSYSTEM

2.1 Background

The Chinese Remainder Theorem was first proposed by Sunzi in the third century, although the complete theorem was given in year 1247 by Dingyi et al. [21]. The theorem has been widely applied in many fields including cryptography, signal processing, vehicular technology, communications, etc. (Chen and Lin [15], Li et al. [49], Wang et al. [88], Wang et al. [92], Wang and Xia [91]).

In this chapter, we introduce several theorems and algorithms aimed at addressing the extended Rabin cryptosystem and the quadratic residue problem. Formally, the quadratic residue problem seeks to find x satisfying $x^2 \equiv a \pmod{n}$, where $a, n \in \mathbb{N}$. Solving this problem entails significant mathematical complexities, particularly when n consists of two large prime numbers, posing a critical challenge in public key cryptography. The computation of square roots of quadratic residues holds relevance in Elliptic Curve cryptography (Mollin [59]) and the Rabin cryptosystem (Rabin [67]). In the Rabin cryptosystem, plaintext encryption involves modular exponentiation, while decryption necessitates extracting the square root of a quadratic residue across two distinct prime modular spaces and subsequently merging the results.

The original Rabin cryptosystem, as described by Rabin, restricts its public key n to be the product of only two distinct large primes, p and q (Rabin [67]). However, this approach lacks theoretical flexibility. Our paper introduces algorithms that extend the Rabin cryptosystem, allowing for a broader range of public keys n composed of any finite number of distinct odd primes raised to various natural number powers, denoted as $n = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$. This enhancement significantly broadens the system's applicability. Additionally, the original Rabin cryptosystem fails to achieve ciphertext indistinguishability. This means that if an adversary possesses the public key n and intercepts the ciphertext c, they can deduce whether c encodes a specific message m by performing encryption on a candidate message. The proposed extended Rabin cryptosystem overcomes this limitation, enhancing security and privacy.

In this chapter, we address the quadratic residue problem modulo n, where $n = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$, utilizing the algorithms presented. Additionally, we develop a generalized algorithm capable of computing the 2^k solutions derived from our method. Theoretical and experimental analyses demonstrate that our proposed algorithms closely match the performance of existing algorithms for cases involving two primes, while simultaneously expanding the generality of the cryptosystem.

The remainder of this chapter is organized as follows: Section 2.2 reviews previous research on the Chinese Remainder Theorem and existing methods for solving the square root of quadratic residues modulo a prime p. Section 2.3 outlines the existing methods and algorithms essential for developing the proposed theorems and algorithms. In Section 2.4, we introduce several theorems applicable to solving systems of linear congruences modulo n, along with corresponding algorithms. Section 2.5 covers the protocol for achieving ciphertext indistinguishability. The subsequent Section 2.6 conducts a thorough experimental analysis, comparing the proposed algorithms with existing ones. These new theorems and algorithms extend the classical Rabin cryptosystem by accommodating a public key composed of an arbitrary finite number of distinct primes. Finally, Section 2.7 concludes the paper and outlines potential future extensions of this work.

2.2 Related Work

In this section, we introduce several pre-existing algorithms designed to address the quadratic residue problem modulo a prime p. We also provide an overview of the Chinese Remainder Theorem and its correspondence to a single linear congruence. This understanding forms the basis for constructing our proposed method to solve for quadratic residues modulo any natural number n.

Over the past 150 years, various algorithms have emerged to tackle the quadratic residue problem modulo a single prime p. Among the pioneers, Tonelli [87] proposed an early efficient algorithm in 1891, subsequently enhanced by Shanks [70] in 1973. Lindhurst [53] demonstrated that Tonelli's algorithm in $\mathbb{F}_p$ typically requires $O((\log p)^3)$ bit operations on average when p and a are chosen randomly. Cipolla [18] introduced the current most efficient algorithm, leveraging Field-Theoretic methods. These algorithms prove valuable in scenarios where no specific condition on p is specified beyond its primality. For instances where $x^2 \equiv a \pmod{p}$, these methods offer solutions, as detailed in Section 2.4. Alternatively, if additional conditions are imposed on p, faster algorithms exist. For instance, when $p \equiv 3 \pmod{4}$, Fermat's Little Theorem can be applied to solve the quadratic residue problem, with a time complexity of $O(\log b)$, making it particularly popular for cryptographic applications (Katz et al. [41]).

The quadratic residue problem finds application in certain types of elliptic curve cryptography, where diverse protocols exist for implementing encryption, decryption, and signing operations (Lopez and Dahab [56]). Typically, these procedures involve evaluating points on an elliptic curve represented by the equation $y^2 = x^3 + ax + b \pmod{p}$, where $p > 2$ and integers a and b are chosen, constituting the Elliptic Curve Discrete Logarithm Problem (ECDLP) (Lopez and Dahab [56]). In elliptic curve cryptography, points on the curve are often stored using point compression techniques (Barreto and Voloch [6]). This involves representing a point (x, y) as (x, β), where β represents a single bit of y. To decompress a point, that is, to recover the value of y from (x, β), it is necessary to compute the square root of a quadratic residue.

The Rabin cryptosystem represents one of the most direct cryptographic applications of the quadratic residue problem. Introduced by Rabin [67] in January 1979, this protocol stands as the inaugural asymmetric cryptosystem to demonstrate that recovering the entire plaintext from the ciphertext is as challenging as factoring integers Rabin [67]. In the original method, Rabin formulated $x^2 \equiv c \pmod{n}$, where c denotes the ciphertext and $n = pq$ with p and q being distinct large primes. One prominent issue with the Rabin cryptosystem is the limited number of solutions to the quadratic residue problem, typically capped at four. To address this challenge, various approaches have been devised, such as incorporating redundancy in the message, transmitting extra bits, or imposing constraints on the message size and factors for n Katz et al. [41].

Takagi [86], Asbullah and Ariffin [3], and Mahad et al. [57] explore strategies for limiting the message size and factors for n. Takagi initially proposed a scheme employing $n = p^2q$, where p and q are distinct large primes, with no restrictions on the message size M. Asbullah and Ariffin introduced a scheme utilizing $n = p^2q$, where p and q are distinct large primes satisfying $p, q \equiv 3 \pmod{4}$, and they restrict the message size M to be within the range $(0, 2^{2n-2})$. Both schemes require the utilization of Algorithms 1, 2, or 3 if p, q do not satisfy $p, q \equiv 3 \pmod{4}$ or

$p, q \equiv 5 \pmod 8$. However, if $p, q \equiv 3 \pmod 4$, applying Fermat's Little Theorem readily solves each congruence.

In addition to its application in cryptography, the Chinese Remainder Theorem finds utility in various fields such as communication, vehicular technology, signal processing, and optical security. In the realm of communication, Chen and Lin [15] proposed an energy-efficient Media Access Control (MAC) address scheme for wireless sensor networks utilizing the Chinese Remainder Theorem. In vehicular technology, Wang et al. [88] employed the Chinese Remainder Theorem ranging method based on dual-frequency measurements , while Li et al. [49] utilized a similar method for phase-detection-based range estimation. In signal processing, Wang and Xia [91] employed the Chinese Remainder Theorem for performance analysis, and Wang et al. [92] discussed the largest dynamic range of a generalized Chinese Remainder Theorem for two integers. Regarding optical security, suggestions have been made for hybrid digital and optical systems to enhance fast and secure cryptography (Javidi et al. [39]). Yatish and Nishchal [98] proposed a triplet of functions for optical cryptosystems, describing a single function on a plaintext input matrix for encryption and two functions for decrypting the ciphertext.

The novel algorithms introduced in Section 2.4 find application in the Rabin cryptosystem, leveraging its direct connection to the quadratic residue problem. These algorithms, detailed in Section 2.4, can employ various methods to compute the square root modulo a prime p. We demonstrate how to generate solutions for the square root of a quadratic residue modulo $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, which yields at most 2^r unique solutions. Traditionally, the Chinese Remainder Theorem is utilized to produce solutions (x modulo n). Subsequently, Section 2.2 outlines existing algorithms for solving the quadratic residue problem modulo a prime p and provides insights into the Chinese Remainder Theorem. Our experimental results involve scenarios with r less than 9 primes, as computational time escalates rapidly when r exceeds 8 with the current implementation. As part of future work, we are exploring a dynamic programming solution for scenarios where r surpasses 8. The algorithms proposed in Section 2.4 stem from extensions of these existing results and algorithms.

2.3 Preliminaries

Tonelli [87] introduced the initial algorithm in 1893, depicted in Algorithm 1. This algorithm represents one of the probabilistic methods for computing the quadratic residue modulo p. There exists a probability of at least 1/2 for a randomly chosen element of $\mathbb{F}_p$ to be a non-square integer. It accepts input a from the congruence $x^2 \equiv a \pmod{p}$ and calculates b such that $b^2 = a$. The time complexity of Tonelli's algorithm is $O((\log p)^4)$. Cipolla [18] introduced an alternative algorithm

---

Algorithm 1 Tonelli's Algorithm

Input: $a \in \mathbb{F}_p^*$, p odd
Output: $b = \sqrt{a}$ on $\mathbb{F}_p^*$

```
  Choose g ∈ F_p^* at random
  if g is a square then
    return -1
  end if
  Let p − 1 = 2^s t, t odd.
  e ← 0.
  for i ← 2 to s do
    if (a g^{−e})^{(p−1)/2^i} ≠ 1 then
      e ← 2^{i−1} + e
    end if
  end for
  h ← a g^{−e}
  b ← g^{e/2} h^{(t+1)/2}
  return b
```

---

in 1903, outlined in Algorithm 2. This algorithm stands as the most efficient known method for computing the square root of a quadratic residue modulo a prime p, without imposing restrictions on the choice of p. Cipolla's approach involves constructing a quadratic extension of $\mathbb{F}_p$ and utilizing an element t whose norm $N(t)$ equals a (Bach and Shallit [4]). The time complexity of Cipolla's algorithm is $O((\log p)^3)$.

The square root modulo n may be determined by using the definition of modulo, $b^2 = a + kn, \exists k, n \in \mathbb{N}, 1 \leq k \leq \left\lfloor \frac{n}{4} \right\rfloor$, as shown in Algorithm 3.

Given the significance of computational speed in cryptographic applications and general computing, it may be prudent to impose constraints on the selection of a prime p, favoring primes where faster algorithms for computing the square root exist. The most widely utilized algorithm in this regard is the one based on Fermat's Little Theorem (Katz et al. [41]). Algorithm 4 provides

Algorithm 2 Cipolla's Algorithm

Input: $a \in \mathbb{F}_p^*$
Output: $b = \sqrt{a}$ on $\mathbb{F}_p^*$

```
Choose t ∈ F_p^* at random
if t^2 − 4a is a square then
   return -1
end if
f ← X^2 − tX + a
b ← X^((p+1)/2) (mod f)
return b
```

Algorithm 3 Algorithm by Exhaustion

Input: $a \in \mathbb{F}_p^*$
Output: $b = \sqrt{a}$ on $\mathbb{F}_p^*$

```
s ← a
while s is not a square do
   s ← s + a
end while
b ← √s
return b
```

the details of this algorithm.

Algorithm 4 Algorithm based on Fermat's Little Theorem

Input: $a \in \mathbb{F}_p^*$ and $p \equiv 3 \pmod 4$

Output: $b = \sqrt{a}$ on $\mathbb{F}_p^*$

```
b ← a^((p+1)/4) (mod p)
return b
```

The first theorem presents the classical method for the Chinese Remainder Theorem (See Kumanduri and Romero [46]).

Theorem 2.3.1. *(Classical Chinese Remainder Theorem) Let $m_1, m_2, ..., m_r$ be pairwise relatively prime integers. Then the simultaneous congruence*

$$\begin{cases} x & \equiv a_1 \pmod{m_1} \\ x & \equiv a_2 \pmod{m_2} \\ & \vdots \\ x & \equiv a_r \pmod{m_r} \end{cases} \tag{1.1}$$

has a unique solution modulo the product $m_1 \times m_2 \times \cdots \times m_r$.

The next theorem is proposed by Nagasaka et al. [62]. It is an improvement to the Chinese Remainder Theorem. This theorem will be the basis for our proposed theorems.

Theorem 2.3.2. *(Improved Algorithm) Under the same assumptions as in the Chinese Remainder, the system of congruences (1.1) is equivalent to the following single linear congruence.*

$$\left(\sum_{i=1}^{r} b_i M_i\right) x \equiv \sum_{i=1}^{r} a_i b_i M_i \pmod{M}$$

where b_i's are arbitrary integers coprime to m_i's, respectively, and $M_i = M/m_i$ for $i = 1, 2, \ldots, r$ with

$$M = \prod_{i=1}^{r} m_i$$

2.4 Proposed Theories and Algorithms

Prior to introducing the proposed algorithms, two original theorems along with their proofs regarding the Chinese Remainder Theorem are established. Subsequently, these results are expanded to address the quadratic residue problem. Lastly, five versatile algorithms are introduced, leveraging the proposed theorems, which can be employed for decryption within the extended Rabin cryptosystem. Additionally, these algorithms can be adapted to address point decompression in an elliptic curve cryptosystem.

Lemma 2.4.1. *Let a system of linear congruences be given:*

$$\begin{cases} x & \equiv a_1 \pmod{m_1} \\ x & \equiv a_2 \pmod{m_2} \\ & \vdots \\ x & \equiv a_r \pmod{m_r} \end{cases}$$

where $\gcd(m_i, m_j) = 1$*, for* $1 \leq i, j \leq r, i \neq j$*. Suppose* $\sum_{i=1}^{n} b_i M_i = 1$*, where* $\gcd(b_i, m_i) = 1$ *and* $M_i = \dfrac{m_1 \times m_2 \times \cdots \times m_r}{m_i}$*. Then the system has a unique solution:*

$$x \equiv \sum_{i=1}^{n} a_i b_i M_i \pmod{M},$$

where $M = m_1 \times m_2 \times \cdots \times m_r$*.*

Proof. From Theorem 2.3.1, the system has a unique solution. From Theorem 2.3.2, the system is equivalent to

$$\left(\sum_{i=1}^{r} b_i M_i \right) x \equiv \sum_{i=1}^{r} a_i b_i M_i \pmod{M}$$

Since $\sum_{i=1}^{r} b_i M_i = 1$, the result follows. □

This result is further expanded to address the quadratic residue problem. Theorem 2.4.1 resolves the general form of the quadratic residue problem by employing the linear congruence derived from Lemma 2.4.1 for any n, where $1 \leq i \leq r, k_i \in \mathbb{N}$, and $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. Corollaries offering special cases of Lemma 2.4.1, which hold significant value in cryptography, are also provided.

Theorem 2.4.1. *Let* $x^2 \equiv a^2 \pmod{n}$*, where* $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$*,* $p_1, ..., p_r$ *are distinct primes, and* $k_1, k_2, ..., k_r \in \mathbb{N}$*. If* $\sum_{i=1}^{r} b_i \dfrac{n}{p_i^{k_i}} = 1$*, and* $(b_1, p_1) = (b_2, p_2) = \cdots = (b_r, p_r) = 1$*, then there are* 2^r *solutions, which are:*

$$x \equiv a \left(\pm b_1 \frac{n}{p_1^{k_1}} \pm b_2 \frac{n}{p_2^{k_2}} \pm \cdots \pm b_r \frac{n}{p_r^{k_r}} \right) \pmod{n}$$

or equivalently,

$$x \equiv x_{j_1,j_2,\cdots,j_r} \equiv a\left[(-1)^{j_1}\left(b_1\frac{n}{p_1^{k_1}}\right)+(-1)^{j_2}\left(b_2\frac{n}{p_2^{k_2}}\right)+\cdots+(-1)^{j_r}\left(b_r\frac{n}{p_r^{k_r}}\right)\right] \pmod{n},$$

where $j_l = 0, 1,\ 1 \le l \le r$.

Proof. First, the square root on both sides is taken to obtain $x \equiv \pm a \pmod{n}$. Then there exists the following system of congruences:

$$\begin{cases} x & \equiv \pm a \pmod{p_1^{k_1}} \\ x & \equiv \pm a \pmod{p_2^{k_2}} \\ & \vdots \\ x & \equiv \pm a \pmod{p_r^{k_r}} \end{cases}$$

Since $\sum_{i=1}^{r} b_i \frac{n}{p_i^{k_i}} = 1$, Lemma 2.4.1 is applied directly. Therefore, the solutions are:

$$x \equiv a\left(\pm b_1\frac{n}{p_1^{k_1}} \pm b_2\frac{n}{p_2^{k_2}} \pm \cdots \pm b_r\frac{n}{p_r^{k_r}}\right) \pmod{n}$$

or equivalently,

$$x \equiv x_{j_1,j_2,\cdots,j_r} \equiv a\left[(-1)^{j_1}\left(b_1\frac{n}{p_1^{k_1}}\right)+(-1)^{j_2}\left(b_2\frac{n}{p_2^{k_2}}\right)+\cdots+(-1)^{j_r}\left(b_r\frac{n}{p_r^{k_r}}\right)\right] \pmod{n},$$

where $j_l = 0, 1,\ 1 \le l \le r$. ∎

Following from Theorem 2.4.1 are two corollaries. Corollary 2.4.2 gives a formula for the solutions of a quadratic residue problem where $n = p_1^{k_1}p_2^{k_2}$, for which there are four solutions.

Corollary 2.4.2. *Let* $x^2 \equiv a^2 \pmod{n}$*, where* $n = p_1^{k_1}p_2^{k_2}$*,* p_1, p_2 *are distinct primes, and* $k_1, k_2 \in \mathbb{N}$*. If* $b_1\frac{n}{p_1^{k_1}} + b_2\frac{n}{p_2^{k_2}} = 1$*, where* $(b_1, p_1) = (b_2, p_2) = 1$*, then*

$$x \equiv a\left(\pm b_1\frac{n}{p_1^{k_1}} \pm b_2\frac{n}{p_2^{k_2}}\right) \pmod{n}$$

Similarly, there exists a formula ($n = p_1^{k_1} p_2^{k_2} p_3^{k_3}$) for the solutions of quadratic residue problem for which there are eight solutions given by Corollary 2.4.3.

Corollary 2.4.3. *Let* $x^2 \equiv a^2 \pmod{n}$*, where* $n = p_1^{k_1} p_2^{k_2} p_3^{k_3}$*,* p_1, p_2, p_3 *are distinct primes, and* $k_1, k_2, k_3 \in \mathbb{N}$.

If $b_1 \frac{n}{p_1^{k_1}} + b_2 \frac{n}{p_2^{k_2}} + b_3 \frac{n}{p_3^{k_3}} = 1$*, where* $(b_1, p_1) = (b_2, p_2) = (b_3, p_3) = 1$*, then*

$$x \equiv a \left(\pm b_1 \frac{n}{p_1^{k_1}} \pm b_2 \frac{n}{p_2^{k_2}} \pm b_3 \frac{n}{p_3^{k_3}} \right) \pmod{n}$$

In situations where the quadratic residue is not a perfect square, various methods can be employed to solve $x^2 \equiv a \pmod{n}$ (1), provided that the congruence (1) is solvable. While Lemma 2.4.1 necessitates a perfect square on the right-hand side, this limitation can be circumvented. By utilizing the prime factorization of $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, Algorithms 1 and 2 can be utilized to find square roots in $\mathbb{F}p_i$, $1 \leq i \leq r$, as elaborated in Section 2.3. Subsequently, each solution can be raised to $\mathbb{F}p_i^{k_i}$ using Hensel's lemma (Rosen [69]). Alternatively, Algorithm 3 can be employed to directly identify the perfect square, or subject to specific constraints discussed in Section 2.3, Algorithm 4 can offer high-performance solutions. Subsequently, Theorem 2.4.4 will yield all 2^r solutions to (1).

Theorem 2.4.4. *Let* $x^2 \equiv a \pmod{n}$*, where* $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$*,* $p_1, ..., p_r$ *are distinct primes, and* $k_1, k_2, ..., k_i \in \mathbb{N}$.

If $\sum_{i=1}^{r} b_i \frac{n}{p_i^{k_i}} = 1$*, where* $(b_1, p_1) = (b_2, p_2) = \cdots = (b_r, p_r) = 1$*, then there are* 2^r *solutions, which are:*

$$x \equiv \pm a_1 b_1 \frac{n}{p_1^{k_1}} \pm a_2 b_2 \frac{n}{p_2^{k_2}} \pm \cdots \pm a_r b_r \frac{n}{p_r^{k_r}} \pmod{n}$$

where $a_i^2 \equiv a \pmod{p_i^{k_i}}$*, or equivalently:*

$$x \equiv x_{j_1, j_2, \cdots, j_r} \equiv (-1)^{j_1} \left(a_1 b_1 \frac{n}{p_1^{k_1}} \right) + (-1)^{j_2} \left(a_2 b_2 \frac{n}{p_2^{k_2}} \right) + \cdots + (-1)^{j_r} \left(a_r b_r \frac{n}{p_r^{k_r}} \right) \pmod{n},$$

where $j_l = 0, 1,\ 1 \leq l \leq r$.

Proof. First reduce the congruence to the following:

$$\begin{cases} x^2 & \equiv a \pmod{p_1^{k_1}} \\ x^2 & \equiv a \pmod{p_2^{k_2}} \\ & \vdots \\ x^2 & \equiv a \pmod{p_r^{k_r}} \end{cases}$$

After reducing modulo $p_i^{k_i}$ in each congruence, the system becomes:

$$\begin{cases} x^2 & \equiv c_1 \pmod{p_1^{k_1}} \\ x^2 & \equiv c_2 \pmod{p_2^{k_2}} \\ & \vdots \\ x^2 & \equiv c_r \pmod{p_r^{k_r}} \end{cases}$$

Note that reducing modulo $p_i^{k_i}$ is not necessary. To obtain a square on the right hand side, any of the methods presented in Section 2.3 can be utilized. Let $a_i^2 \equiv c_i \pmod{p_i^{k_i}}$, then the system becomes:

$$\begin{cases} x^2 & \equiv a_1^2 \pmod{p_1^{k_1}} \\ x^2 & \equiv a_2^2 \pmod{p_2^{k_2}} \\ & \vdots \\ x^2 & \equiv a_r^2 \pmod{p_r^{k_r}} \end{cases}$$

Taking the square root of both sides, the following system is obtained:

$$\begin{cases} x & \equiv \pm a_1 \pmod{p_1^{k_1}} \\ x & \equiv \pm a_2 \pmod{p_2^{k_2}} \\ & \vdots \\ x & \equiv \pm a_r \pmod{p_r^{k_r}} \end{cases}$$

Combining Lemma 2.4.1 and Theorem 2.4.1, the solutions are:

$$x \equiv \pm a_1 b_1 \frac{n}{p_1^{k_1}} \pm a_2 b_2 \frac{n}{p_2^{k_2}} \pm \cdots \pm a_r b_r \frac{n}{p_r^{k_r}} \pmod{n}$$

or equivalently,

$$x \equiv x_{j_1, j_2, \cdots, j_r} \equiv (-1)^{j_1}\left(a_1 b_1 \frac{n}{p_1^{k_1}}\right) + (-1)^{j_2}\left(a_2 b_2 \frac{n}{p_2^{k_2}}\right) + \cdots + (-1)^{j_r}\left(a_r b_r \frac{n}{p_r^{k_r}}\right) \pmod{n},$$

where $j_l = 0, 1$, $1 \leq l \leq r$. □

The subsequent algorithm, presented as a result of Theorem 2.4.1, is designed to compute the square root of a quadratic residue modulo $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. The first algorithm addresses the scenario where $x^2 \equiv a^2 \pmod{n}$, where $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. Given the perfect square corresponding to the quadratic residue $a^2 \pmod{n}$, determinable via Algorithm 4 in Section 2.3, the subsequent algorithm enumerates the 2^r solutions by simplifying the problem to an extended Greatest Common Divisor (GCD) calculation. The extended GCD of two numbers a and b with $\gcd(a, b) = g$ yields a solution $m, n \in \mathbb{Z}$ to the linear Diophantine equation $ma + nb = g$. Similarly, the extended GCD of r integers, $w_1, w_2, \ldots, w_r$, provides a solution $b_1, b_2, \ldots, b_r \in \mathbb{Z}$ satisfying the linear Diophantine equation $b_1 w_1 + b_2 w_2 + \cdots + b_r w_r = g$. An algorithm to compute the extended GCD of r integers is also presented in Algorithm 8 of Section 2.4.

If only the coefficients are required, the algorithm operates in $O(rQ(b) + G(b))$ time, where $Q(b)$ denotes the time complexity for computing the square root of a quadratic residue of b bits, as provided by Algorithms 1 – 4 in Section 2.3. Utilizing Algorithm 4 leads to $Q(b) = O(\log(b))$. Here, $G(b)$ represents the time required for computing the Extended GCD of r numbers, with the given algorithm functioning in $O(rM(b)\log(b))$. Hence, the time complexity for generating the terms for solutions is $O(rM(b)\log(b))$. However, the time complexity for enumerating all solutions is $O(2^r)$. To determine the accurate plaintext, padding bits can be employed and verified during enumeration.

Algorithm 5 Decryption Given $a^2 \pmod n$

Input: $a^2 \pmod n$ and $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

Output: $\boldsymbol{x_{2^r}} = x_1, x_2, \cdots, x_{2^r}$ solutions to square root

$a \leftarrow \pm\sqrt{a^2}$

$b1, b2, \cdots, b_r \leftarrow \text{ExtendedGCD}\left(\frac{n}{p_1^{k_1}}, \frac{n}{p_2^{k_2}}, \cdots, \frac{n}{p_r^{k_r}}\right)$

for $j \leftarrow 1$ to r **do**

  if $\gcd(b_j, p_j) \neq 1$ **then**

    return -1

  end if

end for

return $\boldsymbol{x_{2^r}}$

The output $\boldsymbol{x_{2^r}}$ are the 2^r solutions to the equation $x^2 \equiv a^2 \pmod n$ given by the vector $\boldsymbol{x_{2^r}} = \begin{pmatrix} x_0 & x_1 & \cdots & x_{2^r} \end{pmatrix}$. Algorithm 9 is presented to compute all of the 2^r solutions using recursion, which can be easily implemented.

Algorithm 5 may be extended to account for inputs representing the perfect square associated with each quadratic residue $a_1^2 \pmod{p_1^{k_1}}$, $a_2^2 \pmod{p_2^{k_2}}$, $\ldots$, $a_r^2 \pmod{p_r^{k_r}}$. These values can be computed utilizing various methods such as Tonelli's algorithm, Cipolla's algorithm, or exhaustive enumeration without imposing additional conditions on the primes. Alternatively, Fermat's Little Theorem can be employed if the condition $p_i \equiv 3 \pmod 4$ is satisfied for calculating the perfect squares in modulo $p_1, p_2, \ldots, p_r$ spaces. Subsequently, Hensel's lemma can be employed to elevate these solutions to the modulo spaces $p_1^{k_1}, p_2^{k_2}, \ldots, p_r^{k_r}$.

The output $\boldsymbol{x_{2^r}}$ are the 2^r solutions to the equation $x^2 \equiv a^2 \pmod n$.

Algorithm 7 provides a comprehensive procedure to discover all 2^r solutions, accommodating any quadratic residue a and r primes elevated to arbitrary powers $p_1^{k_1}, p_2^{k_2}, \ldots, p_r^{k_r}$. Various approaches for implementing the PerfectSquare and GenSqrRt functions are available. However, for optimal efficiency, enforcing $p \equiv 3 \pmod 4$ and employing Algorithm 4 in Section 2.3 for PerfectSquare implementation, along with Algorithm 6 in Section 2.4 for GenSqrRt implementation, results in Algorithm 7 and 9 collectively achieving a time complexity of $O(rM(b)log(b) + 2^r)$ when executed

Algorithm 6 Generalized Decryption Algorithm

Input: $a_1^2 \pmod{p_1^{k_1}}$, $a_2^2 \pmod{p_2^{k_2}}$, $\ldots$, $a_r^2 \pmod{p_r^{k_r}}$ perfect squares of quadratic residues and $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

Output: $\boldsymbol{x_{2^r}} = x_1, x_2, \cdots, x_{2^r}$ solutions to square root

for $i \leftarrow 1$ to r **do**

$a_i \leftarrow \pm\sqrt{a_i^2}$

end for

$b1, b2, \cdots, b_r \leftarrow \text{ExtendedGCD}\left(\frac{n}{p_1^{k_1}}, \frac{n}{p_2^{k_2}}, \cdots, \frac{n}{p_r^{k_r}}\right)$

for $j \leftarrow 1$ to r **do**

if $\gcd(b_j, p_j) \neq 1$ **then**

return -1

end if

end for

return $\boldsymbol{x_{2^r}}$

together, where r denotes the number of primes. In instances where $r = 2$, this algorithm exhibits a time complexity of $O(M(b)log(b))$, which is asymptotically comparable to the existing algorithm for the Rabin cryptosystem.

Algorithm 7 General Rabin Decryption Algorithm

Input: $a \in \mathbb{Z}_n$ and $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$

Output: $\boldsymbol{x_{2^r}} = x_1, x_2, \cdots, x_{2^r}$ solutions to square root

for $i \leftarrow 1$ to r **do**

$a_i^2 \leftarrow \text{PerfectSquare}(a,\, p_i^{k_i})$

end for

$\boldsymbol{x_{2^r}} \leftarrow \text{GenSqrRt}(a_1^2,\, a_2^2,\, \ldots,\, a_r^2,\, p_1^{k_1},\, p_2^{k_2},\, \ldots,\, p_r^{k_r})$

return $\boldsymbol{x_{2^r}}$

Utilizing the extended GCD function for two integers available in the GMP Library, which exhibits a time complexity of $O(M(b)log(b))$ (Granlund et al. [26]), where b denotes the number of bits in the numbers being multiplied and $M(b)$ signifies the time required to multiply two b-bit numbers. The ExtendedGCD procedure (Algorithm 8) for r terms involves performing $r - 1$ extended GCD calculations of two numbers, mpz_gcdext(a, b), as outlined in Algorithm 8. Consequently, the time complexity for Algorithm 8 amounts to $O(rM(b)log(b))$.

Algorithm 8 ExtendedGCD($w_1, w_2, \ldots, w_r$)

Input: $b[]$ coefficients of extended GCD, $w[]$, n size of array, gcd g
Output: $b_1, b_2, \cdots, b_i$ coefficients satisfying $b_1w_1 + b_2w_2 + \cdots + b_rw_r = 1$

```
if n < 2 then
  b[0] ← 1
  return
end if
if n (mod 2) = 1 then
  s ← (n + 1)/2
else
  s ← n/2
end if
k ← 0
for i ← 0 to n − 2 by 2 do
  mpz_gcdext(neww[k], b[i], b[i + 1], w[i], w[i + 1])
  k ← k + 1
end for
if n (mod 2) = 1 then
  neww[k] ← w[n − 1]
  b[n − 1] ← 1
end if
if n = 2 then
  g ← neww[k − 1]
end if
ExtendedGCD(g, newb, neww, s)
for i ← 0 to s − 1 do
  b[2i] ← b[2i]newb[i]
  b[2i + 1] ← b[2i + 1]newb[i]
end for
if n (mod 2) = 1 then
  b[n − 1] ← newb[s − 1]
else
  b[2(s − 1)] ← b[2(s − 1)]newb[s − 1]
  b[2(s − 1) + 1] ← b[2(s − 1) + 1]newb[s − 1]
end if
```

To address ExtendedGCD($w_1, w_2, \ldots, w_r$), Algorithm 8 resolves the extended gcd of disjoint pairs from the set $w_1, w_2, \ldots, w_r$. Subsequently, the GCD g_{ij} of each pair, w_i, w_j, serves as the values for the subsequent recursive call of ExtendedGCD. Initially, in the first call of ExtendedGCD, $\frac{r}{2}$ extended GCDs of two numbers are computed, followed by $\frac{r}{4}$ in the subsequent call, and so forth. This process continues until only one value remains, which represents the GCD of $w_1, w_2, \ldots, w_r$,

Algorithm 9 GenerateSolutions

Input: terms of the solutions $a[]$, size of array r
Output: $x[]$ array of 2^r solutions

```
if r = 1 then
    x[0] ← a[0]
    x[1] ← a[1]
    return x
end if
r_y ← 1 ≪ (r − 1)
y ← GenerateSolutions(a, r − 1)
for i ← 0 to r_y do
    x[i] ← y[i] + a[r_y−1]
end for
for i ←r_y to 2 ∗ r_y do
    x[i] ← −x[i − r_y]
end for
return x
```

serving as the termination condition for the function. Consequently, at level i, there are $\frac{r}{2^i}$ calls to mpz_gcdext.

In summary, this results in a geometric series: $\sum_{i=1}^{log_2(r)} \frac{r}{2^i} = r \sum_{i=1}^{log_2(r)} \frac{1}{2^i} = r$. Since each call incurs a time complexity of $O(M(b)log(b))$, Algorithm 8 exhibits a time complexity of $O(rM(b)log(b))$.

Algorithm 9 provides a method for enumerating all 2^r solutions. It can be employed to determine x_{2^r} in Algorithms 5 and 6 with a time complexity of $O(2^r)$. Employing recursion, this algorithm constructs solutions by generating potential combinations where each coefficient can be either 1 or -1 for the sum of r terms, based on combinations for the sum of $r-1$ terms.

2.5 Security Analysis

In this section, we delve into the security implications of the proposed algorithm. While the Rabin cryptosystem traditionally relies on the complexity of integer factorization, it typically employs only two distinct large primes. However, the proposed algorithm offers the flexibility to utilize any fixed number of primes, with the ability to alter this number between sessions, thereby enhancing unpredictability.

One limitation of the classical Rabin cryptosystem lies in its lack of ciphertext indistinguishability. This deficiency stems from the deterministic nature of the encryption process. Specifically, given a public key n, encrypting a message m entails computing $m^2 \equiv c \pmod{n}$. Consequently, if an eavesdropper gains knowledge of n and intercepts an encrypted message c, they can deduce whether c encodes a potential plaintext message $\hat{m}$ by performing $\hat{m}^2 \equiv c_1 \pmod{n}$. Should c_1 match c, it indicates $m = \hat{m}$, thus revealing the plaintext.

Formally, in the context of a message space $m \in a, b$ and a ciphertext c such that $E(m) = c$, ciphertext indistinguishability entails that an adversary cannot ascertain which $m \in a, b$ corresponds to c with a probability exceeding $\frac{1}{2}$. However, in the classical Rabin system, an adversary can deduce which plaintext m corresponds to ciphertext c by encrypting both potential candidates and selecting the correct plaintext based on the comparison with the captured ciphertext.

In the extended Rabin cryptosystem, achieving ciphertext indistinguishability is feasible through a specific approach. To accomplish this, a random number S is incorporated into the public key, and a predetermined number z of the least significant bits of S^z are utilized as padding for the message. This setup establishes a message space where the adversary, despite knowing $m \in a, b$, is confronted with a message actually belonging to $m \in aS^z, bS^z$. Since the adversary lacks knowledge of S, they are unable to discern a candidate message solely from a, b.

Extending the public key entails utilizing two large primes, p and q, alongside a random number S such that $n = pqS$. Generating S involves each participant in the encryption agreeing on a shared secret seed, s_0, and a pseudo-random number generator f. Subsequently, each message is transmitted using a new key derived from p, q, and s_0. The initial message can be encrypted with $S = s_1 = f(s_0)$ and $n_1 = pqs_1$, while the i^{th} message can be encrypted with $s_i = f(s_{i-1})$ and $n_i = pqs_i$. This setup obviates the necessity to share keys $n_1, n_2, \ldots$ for subsequent encryptions.

Consequently, an external eavesdropper intercepting the i^{th} ciphertext, c_i, remains unaware of $n_i = pqs_i$. Since s_i is undisclosed, the z least significant bits s_i^z also remain unknown. Consequently, the eavesdropper cannot determine the appropriate padding to append to the candidate messages. However, both intended parties possess knowledge of the initial seed s_0 and the function f, allowing them to ascertain n_i for each message, given that the sequence $s_1, s_2, \ldots$ is deterministic based on s_0.

Notably, recognizing that any given random number s_i possesses a prime factorization, $s_i =$

$p_{s_i1}^{k_{s_i1}} p_{s_i2}^{k_{s_i2}} \cdots p_{s_ir}^{k_{s_ir}}$, it follows that $n_i = pqs_i = pqp_{s_i1}^{k_{s_i1}} p_{s_i2}^{k_{s_i2}} \cdots p_{s_ir}^{k_{s_ir}}$. Consequently, the algorithms proposed in Section 2.4 can be employed to decrypt a ciphertext c_i using the key n_i.

With a public key $n = pq$ and a shared secret (s_0, f), alongside a message space $m \in a, b$, Alice transmits an encrypted message c_i to Bob through the following steps. Initially, Alice calculates $s_i = f(s_{i-1})$ and constructs the key $n_i = pqs_i$. She then encrypts the plaintext m_i by incorporating appended padding bits s_i^z, employing the equation $m_i^2 \equiv c_i \pmod{n_i}$.

Bob independently computes s_i and n_i using the same mechanism as Alice, thus eliminating the need for Alice to transmit n_i over the public channel. Consequently, an adversary in possession of n who intercepts a ciphertext corresponding to the i^{th} encryption, denoted as c_i, cannot discern m_i by encrypting candidate messages $\hat{m}$ with $\hat{m} \equiv \hat{c} \pmod{n}$ and subsequently comparing $\hat{c}$ to c_i. This challenge arises because c_i wasn't encrypted utilizing the public key n, and the message m_i was augmented with padding bits selected from s_i.

Since the selection of s_i is random, the adversary faces impracticality in guessing the key n_i, even if n is known. Similarly, it proves daunting for the adversary to deduce the padding utilized, s_i^z, for the message m_i, given that, as long as z exceeds one, there exist more than two potential choices for s_i^z.

2.6 Experimental Analysis

This section presents the experimental evaluation conducted on the algorithms outlined in Section 2.4. These algorithms are applied within the context of the Rabin cryptosystem due to its direct relevance to the quadratic residue problem. For the experiments discussed herein, Algorithm 7 is utilized, where the PerfectSquare function is implemented employing Fermat's Little Theorem (Algorithm 4 in Section 2.3), and the GenSqrRt function is implemented utilizing Algorithm 6 from Section 2.4.

The experimental evaluation encompasses three distinct experiments. Firstly, in the initial experiment, $n = pq$ remains fixed to enable a direct comparison between the proposed algorithm and the existing decryption algorithm employed within the Rabin cryptosystem. The existing algorithm is derived from the classical Chinese Remainder Theorem (Theorem 2.3.1).

Secondly, the subsequent experiment involves utilizing two primes while varying the number of bits in each prime. This variation is employed to assess the runtime of our algorithm in comparison with the existing decryption algorithm utilized within the Rabin cryptosystem.

Lastly, the third experiment examines the runtime of Algorithm 6 from Section 2.4 as a function of the number of primes in the prime factorization of n. This experiment aims to provide insights into the algorithm's scalability concerning the number of primes utilized.

2.6.1 Specifications

All the algorithms employed in our experiments are coded in C++11 utilizing the g++ compiler version 7.3.0. Compiler optimizations are applied using the -O3 flag for all tests. Given the size of the integers involved in the calculations, the GNU MP Bignum Library (GMP) is utilized for handling integers with arbitrary precision (Granlund et al. [26]).

To measure the runtime of each test, the high-resolution clock provided by the C++11 Chrono library is employed. The experiments are conducted within a 64-bit Ubuntu 18.04 LTS virtual machine hosted on a Windows 10.0.17134 operating system, using VirtualBox 6.0.4 r128413 (Qt5.6.2). The virtual machine is allocated 2 hardware cores (equivalent to 4 logical cores) and 40,000 MB of RAM, with VT-x/AMD-V, nested paging, and KVM paravirtualization enabled.

The host computer used for running the tests is equipped with an Intel Xeon E5-1630 v4 processor clocked at 3.70 GHz, featuring 4 physical cores and 8 logical processors, along with 64 GB of RAM.

2.6.2 Data Description

The verification of the proposed method involved six datasets sourced from the UCI Machine Learning Repository (Frank [23]). These datasets vary in size, ranging from 6.2 KB to 4.0 MB. The selection of datasets of different sizes was deliberate to showcase the algorithm's performance across diverse scenarios. For instance, one potential application could involve employing the proposed algorithm to encrypt symmetric keys. Notably, the type and content of the datasets are immaterial to the encryption-decryption process.

The Balance Scale dataset (Balance) represents the smallest dataset utilized in our experiments, comprising 6250 bytes of character-encoded data. It comprises 625 instances featuring four attributes used for classifying scales as balanced, tipped to the right, or tipped to the left.

Moving on to the Computer Hardware dataset (Machine), it contains 8726 bytes of character-encoded data, encompassing 209 instances with nine attributes utilized to describe the relative performance of CPU chips.

The Car Evaluation dataset (Car) is more extensive, containing 51867 bytes of character-encoded data. It includes 1728 instances featuring six attributes used to evaluate various car models.

Next, the Chess dataset (krkopt) comprises 531,806 character-encoded bytes, consisting of 28,056 instances featuring six classifying attributes. This dataset is designed to evaluate common chess endgame positions involving a white king and rook against a lone black king.

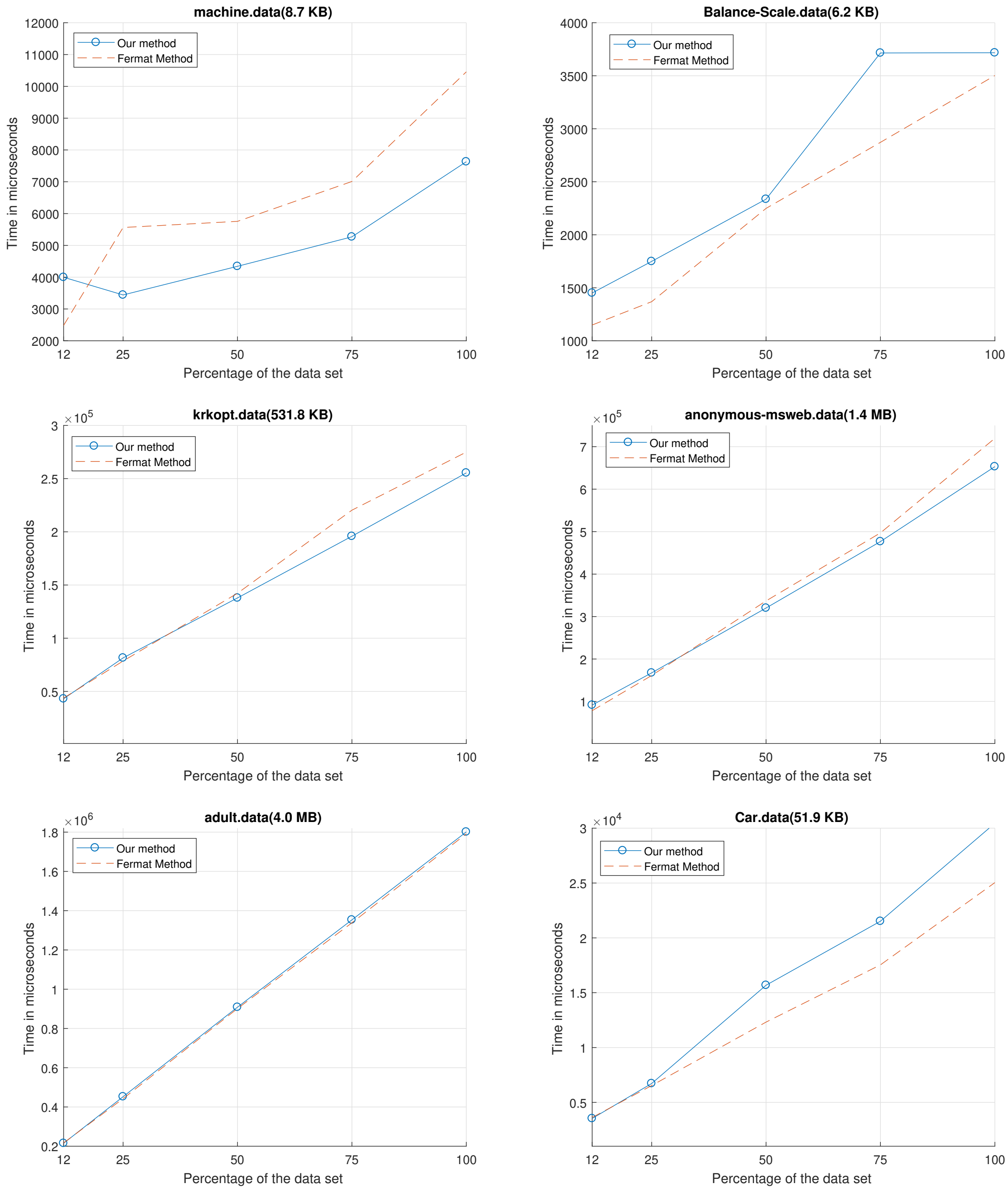
The Anonymous Microsoft Web Data dataset (Anonymous) is notably large, encompassing 1,423,098 character-encoded bytes. It comprises 37,711 instances and 294 attributes compiled from user visits to the www.microsoft.com website.

Lastly, the Adult dataset (Adult) represents the largest dataset tested, containing 3,974,305 character bytes. It consists of 48,842 instances featuring 14 attributes utilized for classifying income brackets based on census information.

These datasets were selected based on their varying sizes, accessibility, and ease of use, rather than their specific application to machine learning. Each dataset contains numerous repeated character strings. However, during encryption, blocks of plaintext relative to the key size are encrypted. Given that the block size exceeds that of a single word, it is improbable for two blocks to be identical in the experiments.

2.6.3 Experimental Results

The experiments partition the large datasets into segments or chunks. The size of each block is proportional to the key size, ensuring it remains sufficiently small to prevent degenerate cases during encryption. These datasets were selected with the aim of illustrating discernible patterns in the algorithm's performance, rather than focusing solely on the application of encrypting symmetric keys. A degenerate case in encryption occurs when two or more plaintexts are mapped to the same ciphertext. Both algorithms employed in the experiments follow the same encryption process.



(e) Figure 2.1e (f) Figure 2.1f

Figure 2.1: Average run-time of 10 trials with varying size of plaintext shows the proposed method is competitive to the existing method when $n = pq$.

Figures 2.1a to 2.1f depict the outcomes of the first experiment, illustrating the average runtime in microseconds of each algorithm relative to the percentage of the dataset encrypted. These figures reveal that the proposed algorithm exhibits linear growth concerning the size of the plaintext. Specifically, Fig. 2.1a demonstrates that the proposed algorithm outperforms Fermat's method by approximately 28%. Conversely, Fig. 2.1e indicates that the proposed algorithm performs comparably to Fermat's method in terms of runtime.

However, Fig. 2.1f stands out as the sole instance where the proposed algorithm did not surpass the performance of the existing method. This observation can be attributed to the relatively small size of the dataset used in Fig. 2.1f, which may have amplified the overhead associated with the proposed algorithm, leading to slower performance. Nevertheless, in larger datasets, this overhead is negligible.

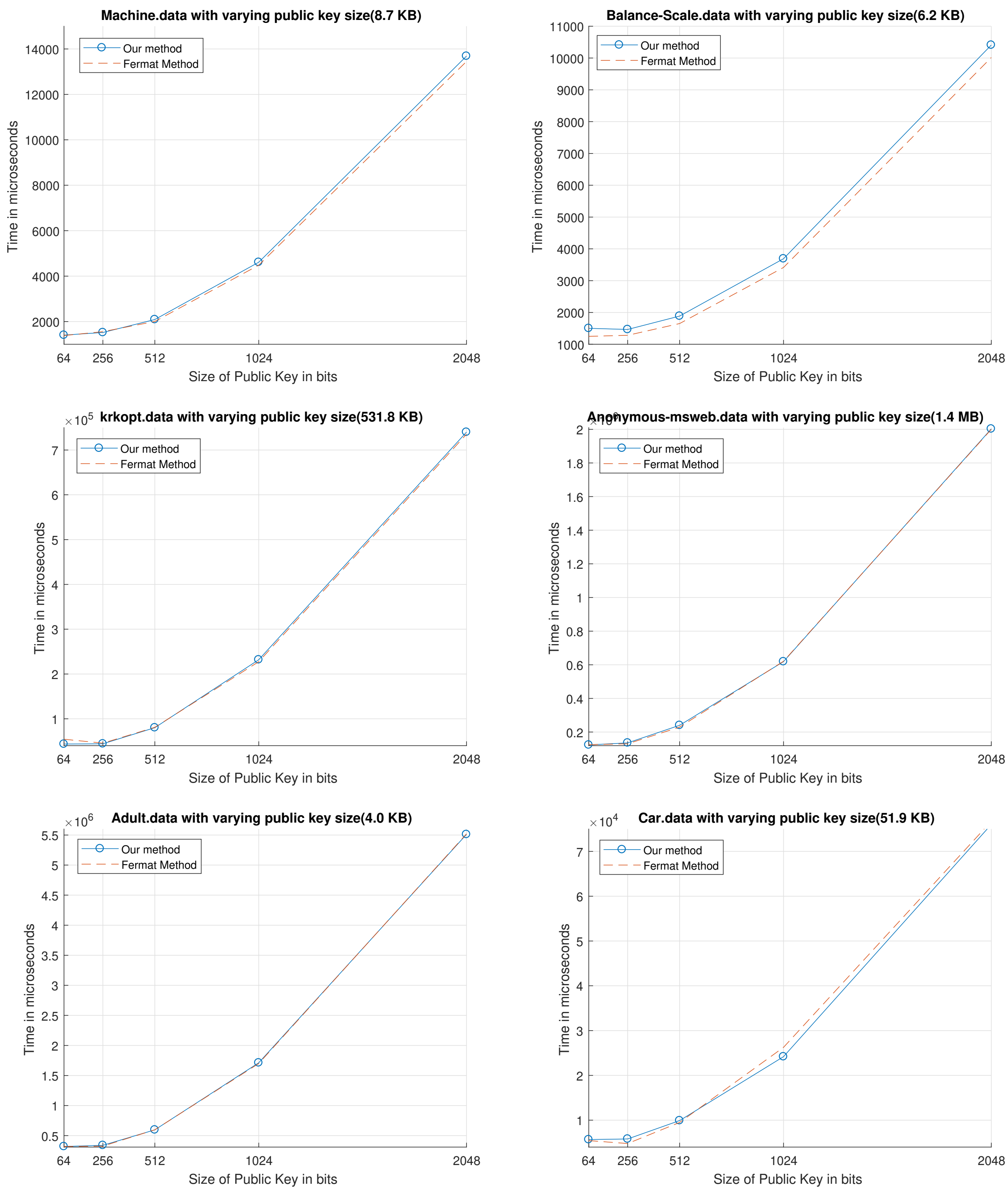
The comparative results are summarized in Table 2.1, with the superior outcomes highlighted in bold for clarity. These results underscore that the proposed algorithm can deliver comparable speed across various dataset sizes compared to the existing algorithm, while simultaneously expanding the cryptosystem's capabilities to accommodate an unlimited number of primes. Figures 2.2a to

Data Set	% of the data set	12	25	50	75	100
Machine	Fermat Method (μs)	2484.7	5562.4	5755.5	7009.1	10456
	Proposed Method (μs)	3997.7	**3441.6**	**4342.3**	**5268.5**	**7631**
krkopt	Fermat Method (μs)	43758	78560.7	142292.5	220350.7	274980.9
	Proposed Method (μs)	**43070**	81497.5	**137878.6**	**195869.9**	**255420.6**
Car	Fermat Method (μs)	3624	6534	12320	17540	25037
	Proposed Method (μs)	**3532**	6721	15682	21519	30448
Balance	Fermat Method (μs)	1451	1749	2336	3715	37177
	Proposed Method (μs)	**1149**	**1369**	**2249**	**2872**	**3502**
Anonymous	Fermat Method (μs)	78375	161216	336841	497071	719737
	Proposed Method (μs)	91606	167405	**320115**	**476634**	**653084**
Adult	Fermat Method (μs)	217150	442520	901445	1338043	1791286
	Proposed Method (μs)	**215258**	452492	908394	1353018	1801056

Table 2.1: Performance comparison in microseconds highlights the superiority of the proposed method.

2.2f depict the outcomes of the second experiment, which compares the proposed algorithm against Fermat's method across varying public key sizes. The experiment demonstrates that the proposed algorithm exhibits a time complexity similar to that of the existing two-prime algorithm utilized in the Rabin cryptosystem, regardless of the key size. Notably, the experimental results for the proposed algorithm align consistently with those of the existing methods.

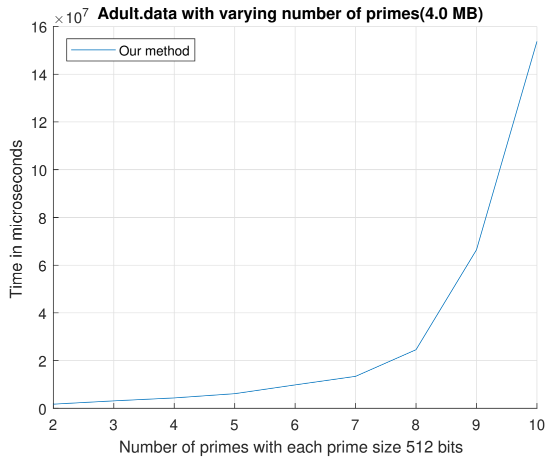
Figure 2.3a illustrates the final experiment, where the proposed algorithm is assessed with a



(e) Figure 2.2e

(f) Figure 2.2f

Figure 2.2: Average run time of 10 trials with varying key size shows the proposed method is competitive when $n = pq$.



(a) Figure 2.3a

Figure 2.3: The average run time is not greatly affected with less than 9 primes in the factorization of n.

public key composed of varying numbers of primes. This experiment showcases the extensibility of the Rabin cryptosystem, allowing for an unlimited number of primes to construct the shared public key. Each prime in this experiment is set to a size of 512 bits.

The results depicted in Figure 2.3a indicate that utilizing eight or fewer primes in the prime factorization of the public key results in negligible performance trade-offs. Consequently, the proposed algorithm for achieving indistinguishability, as outlined in Section 2.5, can be constructed with a random number s derived from a combination of up to eight primes, each with 512 bits, amounting to a range of $[0, 2^{4096} - 1]$. This approach ensures the efficiency of the unmodified extended Rabin cryptosystem is maintained.

The graph displayed exhibits exponential growth in the number of primes due to the algorithm utilized to generate the 2^r solutions to the quadratic residue problem, where r represents the number of primes. Although the proposed algorithm employed to enumerate the solutions operates with a time complexity of $O(2^r)$, it is feasible to implement an algorithm with polynomial time complexity using dynamic programming techniques.

2.7 Conclusion

This chapter introduces several theorems and algorithms pertaining to the Chinese Remainder Theorem and the extended Rabin cryptosystem. The experimental analysis conducted on these proposed algorithms, when applied to the Rabin cryptosystem, reveals a notable improvement in performance compared to existing algorithms for the Rabin cryptosystem. Additionally, the proposed algorithms offer enhanced support for encryption across any ring of integers modulo n, accommodating an arbitrary number of distinct primes in its prime factorization while preserving the elegance of a solution utilizing only two primes.

Moreover, the study demonstrates how this expanded generality can facilitate the provision of ciphertext indistinguishability to the extended Rabin cryptosystem. Furthermore, by augmenting the key size, it becomes possible to construct a larger message space, thereby mitigating the fragmentation of large plaintexts that typically result in an increased number of decryption operations within smaller message spaces.

The algorithm designed to enumerate all 2^r solutions with polynomial time complexity remains a subject for future exploration and refinement.

The publication can be seen in Zhan et al. [99].

CHAPTER 3

ON ALGORITHMS FOR COMPUTING SUMS OF POWERS OF ARITHMETIC PROGRESSIONS

3.1 Background

The sums of powers of arithmetic progressions is an important topic in computational combinatorics and number theory. This sum is of the form

$$S_{p,a,d}(n) = a^p + (a+d)^p + \cdots + (a+(n-1)d)^p = \sum_{j=0}^{n-1}(a+jd)^p, \tag{3.1}$$

where $n \geq 1$, p is a non-negative integer, and a and d are complex numbers with $d \neq 0$. Several formulae and algorithms using different special functions and numbers have been developed to compute this sum (Bounebirat et al. [9], Shiue et al. [80], Xiong et al. [97], Laissaoui and Rahmani [47]).

The Faulhaber's formula arises as the special case of the sum when $a = d = 1$ in (3.1):

$$S_{p,1,1}(n) = \sum_{j=1}^{n} j^p = 1^p + 2^p + 3^p + \cdots + n^p. \tag{3.2}$$

We obtain the famous formula when $p = 1$:

$$S_{1,1,1}(n) = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}.$$

The classical theorem of Faulhaber states that the sums of odd powers

$$1^{2m-1} + 2^{2m-1} + \cdots + n^{2m-1}$$

can be expressed as a polynomial of the triangular numbers $T_n = \dfrac{n(n+1)}{2}$ (Knuth [42]). Chen et al. [13] observed that the classical Faulhaber's theorem on sum of odd powers also holds for arbitrary arithmetic progressions, namely, the odd power sums of any arithmetic progression

$$a, a+b, \cdots, a+(n-1)b$$

can be expressed as a polynomial in $na + \dfrac{n(n+1)b}{2}$.

Finding the optimal formula and algorithm for calculating (3.1) continues to intrigue both the mathematics and computer science realms. Laissaoui and Rahmani [47] provided a clear-cut formula expressing the polynomial in terms of n, leveraging Stirling numbers of the second kind and binomial coefficients, as detailed in the subsequent section.

In 2013, Xiong et al. [97] introduced a technique for computing such a sum based on general Eulerian numbers. However, utilizing generalized Eulerian numbers might not be optimal due to the increased computational time required. Several other researchers have also explored results utilizing generalized Eulerian numbers Lehmer [48], Pita-Ruiz [66]. This study aims to propose a novel approach using classical Eulerian numbers for comparison with Xiong et al.'s method employing general Eulerian numbers. Specifically, we conduct an algorithmic analysis of their method to determine its theoretical time complexity.

In this chapter, we classify our findings into two groups, sums obtained using Stirling number of the second kind and those obtained using classical Eulerian numbers. Within each category, we provide detailed algorithms and analyses. Additionally, we present experimental findings related to these algorithms.

3.2 Preliminaries

3.2.1 Stirling Number of the Second Kind

Since the binomial coefficients are involved, it is important to note that Pascal's identity (Aigner [1]),

$$\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k},$$

can be written as

$$\binom{r}{k} = \binom{r+1}{k+1} - \binom{r}{k+1}, \tag{3.3}$$

where r may not be a positive integer.

Charalambides [12] gave the expansion of t^p in terms of the Stirling numbers of the second kind and the binomial coefficients, written as follows:

$$t^p = \sum_{j=1}^{p} j!S(p,j)\binom{t}{j}. \tag{3.4}$$

The Stirling numbers of the second kind, denoted by $S(p,k)$, count the number of ways to partition a set of p labelled objects into k nonempty unlabelled subsets (Charalambides [12]). The explicit formula can be written as

$$S(p,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-i)^p, \tag{3.5}$$

where $\binom{k}{j}$ denotes the binomial coefficients. The Stirling numbers of the second kind satisfy the following recurrence relation

$$S(p+1,k) = kS(p,k) + S(p,k-1), \tag{3.6}$$

with $S(0,0) = 1$ and $S(p,0) = S(0,k) = 0$.

3.2.2 Classical and General Eulerian Number

The classical Eulerian number $\left\langle {p \atop j} \right\rangle$ is the number of permutations of the numbers 0 to p in which exactly j elements are greater than the previous element.

The classical Eulerian numbers are defined as the coefficients of $\binom{x+p-j}{p}$, $j = 0, 1, \cdots, n$ in the factorial expansion of x^n, namely,

$$x^p = \sum_{j=0}^{p} \left\langle {p \atop j} \right\rangle \binom{x+j-1}{p}, \quad p = 0, 1, \cdots, \tag{3.7}$$

which is called Worpitzky's identity (Carlitz [10], Charalambides [12], Worpitzky [96]). For example, the values of $\left\langle {p \atop j} \right\rangle$ for $1 \le p \le 5$ are given as follows (Charalambides [12]):

p \ j	0	1	2	3	4
1	1				
2	1	1			
3	1	4	1		
4	1	11	11	1	
5	1	26	66	26	1

Table 3.1: Values of Eulerian numbers for $1 \le p \le 5$.

Xiong et al. [97] defined the general Eulerian numbers $A_{p,j}(a, d)$ with the following recurrence relation as $A_{0,-1} = 1$, $A_{p,j} = 0$ ($j \ge p$ or $j \le -2$) and

$$A_{p,j}(a, d) = (-a + (j+2)d)A_{p-1,j}(a, d) + (a + (p-j-1)d)A_{p-1,j-1}(a, d). \tag{3.8}$$

$$(0 \le j \le p-1).$$

The binomial expansion of $\binom{x}{p}$ can be rewritten using the Pochhammer symbol $(x)_p$, or falling factorial, as follows:

$$\binom{x}{p} = \frac{(x)_p}{p!}, \tag{3.9}$$

where $(x)_p = x(x-1)\cdots(x-p+1)$. We will use this in our main results.

Next, we present two lemmas for the explicit forms of Eulerian numbers and general Eulerian numbers, which we will use in the experiments.

Lemma 3.2.1. *The classical Eulerian numbers* $\left\langle {p \atop j} \right\rangle$ *are given explicitly by the sum*

$$\left\langle {p \atop j} \right\rangle = \sum_{i=0}^{j} (-1)^i \binom{p+1}{i} (j-i)^p. \tag{3.10}$$

This lemma is proved in many manuscripts (Charalambides [12], Comtet [19], Hsu and Shiue [34]).

Lemma 3.2.2. *The general Eulerian numbers* $A_{p,j}(a,d)$ *are given explicitly by the sum*

$$A_{p,j}(a,d) = \sum_{i=0}^{j+1} (-1)^i \binom{p+1}{i} [(j+2-i)d - a]^p. \tag{3.11}$$

This lemma is proved by Xiong et al. [97].

Xiong et al. [97] used the general Eulerian numbers to compute (3.1) in the following formula:

$$\sum_{k=1}^{n} (a + d(k-1))^p = \sum_{j=-1}^{p-1} A_{p,j}(a,d) \binom{n+j+1}{p+1}. \tag{3.12}$$

A comparison of (3.12) with our results is given in Section 3.5.

3.3 Main Results

In this section, we categorize our results into two subsections.

3.3.1 Stirling Number of the Second Kind

We give an elementary approach to prove the following theorem of Laissaoui and Rahmani [47].

Theorem 3.3.1. *(Laissaoui and Rahmani [47]). Let* a *and* d *be complex numbers with* $d \neq 0$, p *a non-negative integer, and* n *a positive integer. Then the sum of the* p^{th} *powers of the first* n *terms in the arithmetic progression with first term* a *and common difference* d *can be expressed as*

$$S_{p,a,d}(n) = d^p \sum_{k=0}^{p} k! S(p,k) \left[\binom{n + \frac{a}{d}}{k+1} - \binom{\frac{a}{d}}{k+1} \right], \tag{3.13}$$

where $S(p,k)$ *denotes the Stirling number of the second kind.*

Proof. Using (3.4), let $t = (k-1) + \dfrac{a}{d}$, then

$$\left((k-1) + \frac{a}{d}\right)^p = \sum_{j=1}^{p} j!S(p,j)\binom{(k-1)+\frac{a}{d}}{j}.$$

Multiplying d^p to both sides gives

$$(a + (k-1)d)^p = d^p \sum_{j=1}^{p} j!S(p,j)\binom{(k-1)+\frac{a}{d}}{j}.$$

Next, summing both sides from $k = 1$ to n gives

$$\sum_{k=1}^{n} \big(a + (k-1)d\big)^p = d^p \sum_{j=1}^{p} j!S(p,j) \sum_{k=1}^{n} \binom{(k-1)+\frac{a}{d}}{j}. \tag{3.14}$$

The binomial term on the right hand side of (3.14) needs to be investigated as k varies from 1 to n. Pascal's identity given in (3.3) is utilized.

For $k = 1$,

$$\binom{(k-1)+\frac{a}{d}}{j} = \binom{\frac{a}{d}}{j} = \binom{\frac{a}{d}+1}{j+1} - \binom{\frac{a}{d}}{j+1}.$$

For $k = 2$,

$$\binom{(k-1)+\frac{a}{d}}{j} = \binom{\frac{a}{d}+1}{j} = \binom{\frac{a}{d}+2}{j+1} - \binom{\frac{a}{d}+1}{j+1}.$$

Inductively, it can be shown that for $k = n$,

$$\binom{(n-1)+\frac{a}{d}}{j} = \binom{\frac{a}{d}+n}{j+1} - \binom{\frac{a}{d}+(n-1)}{j+1}.$$

Hence,

$$\sum_{k=1}^{n} \binom{(k-1)+\frac{a}{d}}{j} = \binom{\frac{a}{d}+n}{j+1} - \binom{\frac{a}{d}}{j+1}. \tag{3.15}$$

Combining the results of (3.14) and (3.15) gives

$$S_{p,a,d}(n) = d^p \sum_{j=1}^{p} j! S(p,j) \left[\binom{n + \frac{a}{d}}{j+1} - \binom{\frac{a}{d}}{j} \right],$$

which is the desired result. Note: the beginning index starts at $j = 1$ because $S(p, 0) = 0$. □

The next corollary simplifies (3.13) to reduce the actual-run-time, which is discussed in more details Section 3.4.

Corollary 3.3.2. *Under the same assumptions as in Theorem 3.3.1, the sums of pth powers of the first n terms in the arithmetic progression with first term a and common difference d can be expressed as*

$$S_{p,a,d}(n) = \frac{d^p}{k+1} \sum_{k=1}^{p} S(p,k) \left[\left(n + \frac{a}{d}\right)_{k+1} - \left(\frac{a}{d}\right)_{k+1} \right]. \tag{3.16}$$

where S(p,k) denotes the Stirling number of the second kind, and and $(x)_n$ denotes the falling factorial $(x)_n = x(x-1)\cdots(x-(n-1))$.

Proof. Use (3.9) to simplify the binomial coefficients in (3.13). □

3.3.2 Classical Eulerian Number

Theorem 3.3.3. *Let a and d be complex numbers with $d \neq 0$, p a non-negative integer, and n a positive integer. Then the sum of the p^{th} powers of the first n terms in the arithmetic progression with first term a and common difference d can be expressed as*

$$\sum_{k=1}^{n} [a + d(k-1)]^p = \frac{d^p}{(p+1)!} \sum_{j=0}^{p} \left\langle {p \atop j} \right\rangle \left[\left(\frac{a}{d} + j + n - 1\right)_{p+1} - \left(\frac{a}{d} + j - 1\right)_{p+1} \right]. \tag{3.17}$$

where $\left\langle {p \atop j} \right\rangle$ denotes the classical Eulerian numbers.

Proof. Given

$$x^p = \sum_{j=0}^{p} \left\langle {p \atop j} \right\rangle \binom{x+j-1}{p}, \tag{3.18}$$

the substitution $x = (k-1) + \frac{a}{d}$ results in

$$\left[(k-1) + \frac{a}{d}\right]^p = \sum_{j=0}^{p} \left\langle {p \atop j} \right\rangle \binom{(k-1) + \frac{a}{d} + j - 1}{p}. \tag{3.19}$$

Multiplying both sides by d^p and summing over k leads to

$$\sum_{k=1}^{n} [a + d(k-1)]^p = d^p \sum_{j=0}^{p} \left\langle {p \atop j} \right\rangle \sum_{k=1}^{n} \binom{(k-1) + \frac{a}{d} + j - 1}{p}. \tag{3.20}$$

Now, consider just $\sum_{k=1}^{n} \binom{(k-1) + \frac{a}{d} + j - 1}{p}$. Expanding the right hand side of (3.20) results in

$$\sum_{k=1}^{n} \binom{(k-1) + \frac{a}{d} + j - 1}{p} = \binom{\frac{a}{d} + j - 1}{p} + \binom{\frac{a}{d} + j}{p} + \cdots + \binom{n - 2 + \frac{a}{d} + j}{p}.$$

Now using Pascal's identity $\binom{n}{p} = \binom{n-1}{p-1} + \binom{n-1}{p}$, the right hand side becomes

$$\binom{\frac{a}{d} + j}{p+1} - \binom{\frac{a}{d} + j - 1}{p+1} + \binom{1 + \frac{a}{d} + j}{p+1} - \binom{\frac{a}{d} + j}{p+1} + \cdots + \binom{n - 1 + \frac{a}{d} + j}{p+1} - \binom{n - 2 + \frac{a}{d} + j}{p+1}.$$

We see that the sum telescopes, so we have the following result

$$\sum_{k=1}^{n} \binom{(k-1) + \frac{a}{d} + j}{p} = \binom{n - 1 + \frac{a}{d} + j}{p+1} - \binom{\frac{a}{d} + j - 1}{p+1}. \tag{3.21}$$

Utilizing the identity (3.9) we arrive at the following

$$\binom{n - 1 + \frac{a}{d} + j}{p+1} - \binom{\frac{a}{d} + j - 1}{p+1} = \frac{(\frac{a}{d} + j + n - 1)_{p+1} - (\frac{a}{d} + j - 1)_{p+1}}{(p+1)!}. \tag{3.22}$$

Plugging (3.22) back into (3.20) gives us

$$\sum_{k=1}^{n} [a + d(k-1)]^p = \frac{d^p}{(p+1)!} \sum_{j=0}^{p} \left\langle {p \atop j} \right\rangle \left[\left(\frac{a}{d} + j + n - 1\right)_{p+1} - \left(\frac{a}{d} + j - 1\right)_{p+1} \right].$$

□

3.4 Algorithms and Analysis

First, we give the algorithm to compute binomial coefficient $\binom{n}{k}$. The time complexity is $O(nk)$.

Algorithm 10 Algorithm for computing the binomial coefficient using dynamic programming

---

Input: Non-negative integers n, k, $k \leq n$.

Output: Binomial coefficient $\binom{n}{k}$

$C \leftarrow$ [[0 for _ in range($k+1$)] for _ in range($n+1$)]
for i in range($n+1$) **do**
  for j in range($\min(i,k)+1$) **do**
    if $j == 0$ or $j == 1$ **then**
      $C[i][j] = 1$
    else
      $C[i][j] = C[i-1][j-1] + C[i-1][j]$
    end if
  end for
end for
Return $C[n][k]$

3.4.1 Stirling Number of the Second Kind

Now, we give the algorithm to compute the Stirling number of the second kind. It has a time complexity of $O(nk)$

Algorithm 11 Algorithm for computing Stirling numbers of the second kind using dynamic programming

Input: Non-negative integers n, k, $k \leq n$.

Output: Stirling number of the second kind $S(n, k)$

$S \leftarrow$ [[0 for _ in range($k+1$)] for _ in range($n+1$)]
for i in range($n+1$) **do**
    $S[i][0] = 0$
    $S[i][i] = 1$
end for
for i in range($1, n+1$) **do**
    for J in range($1, \min(i, k)+1$) **do**
        $S[i][j] = j \times S[i-1][j] + S[i-1][j-1]$
    end for
end for
Return $S[n][k]$

Finally, we can compute the sum using Stirling number of the second kind:

Algorithm 12 Computing the sums using Stirling number of the second kind

Input: $a, d \in \mathbb{C}$, $d \neq 0$, p non-negative integer, and $a \in \mathbb{N}$.

Output: Sum of arithmetic progression $S_{p,a,d}(n)$.

Initialize Sum $M = 0$
for j in range($1, p+1$) **do**
    Compute $m_1 = j!$
    Compute $m_2 = S(p, j)$ using Algorithm 11
    Compute $m_3 = \binom{n+\frac{a}{d}}{j+1} - \binom{\frac{a}{d}}{j}$ using Algorithm 10
    $M = M + m_1 \times m_2 \times m_3$
end for
Compute $S_{p,a,d}(n) = d^p \times M$
Return $S_{p,a,d}(n)$

Computing $m_1 = j!$ requires $O(j)$ operations, but since j ranges from 1 to p, the total complexity for this step is $O(p^2)$. Computing $m_2 = S(p, j)$ is $O(pj) = O(p^2)$ since j is the running index. The binomial coefficients in m_3 can be computed in $O(nj) = O(n^2)$. In total, the time complexity is $O(p^3 + pn^2)$.

Next, we give the corresponding algorithm to Corollary 3.3.2.

Algorithm 13 Algorithm to compute the sum using Corollary 3.3.2

Input: $a, d \in \mathbb{C}$, $d \neq 0$, p non-negative integer, and $a \in \mathbb{N}$.

Output: Sum of arithmetic progression $S_{p,a,d}(n)$.

Initialize $M = 0$

for k in range$(1, p+1)$ **do**

  Compute $m_1 = S(p, k)$ using Algorithm 11

  Compute $m_2 = \left(n + \frac{a}{d}\right)_{k+1} - \left(\frac{a}{d}\right)_{k+1}$

  $M = M + m_1 \times m_2$

end for

Compute $S_{p,a,d}(n) = \frac{d^p}{k+1}$

Return $S_{p,a,d}(n)$

In this algorithm, we have the time complexity $O(p)$ for the for-loop, $O(p^2)$ for the Stirling number of the second kind, and $O(p)$ for computing the Pochhammer symbol. In total, we have $O(p)(O(p^2) + O(p)) = O(p^3)$.

3.4.2 Classical Eulerian Number

Now, we give the algorithm for computing the classical Eulerian numbers. It has a time complexity of $O(n^2)$.

Algorithm 14 Algorithm to compute the classical Eulerian numbers

Input: Non-negative integers n, k, $k \leq n$.

Output: Classical Eulerian numbers $\left\langle {n \atop k} \right\rangle$.

$E \leftarrow [[0 \text{ for \_ in range}(k+1)] \text{ for \_ in range}(n+1)]$
for i in range$(n+1)$ **do**
  for j in range$(\min(i,k)+1)$ **do**
    if $j == 0$ or $j == 1$ **then**
      $E[i][j] = 1$
    else
      $E[i][j] = (i-j) \times E[i-1][j-1] + (j+1) \times E[i-1][j]$
    end if
  end for
end for
Return $E[n][k]$

Finally, we give the algorithm to compute the sum using classical Eulerian numbers.

Algorithm 15 Algorithm to compute the sum using classical Eulerian numbers

Input: $a, d \in \mathbb{C}$, $d \neq 0$, p non-negative integer, and $a \in \mathbb{N}$.

Output: Sum of arithmetic progression $S_{p,a,d}(n)$.

Initialize $M = 0$
for j in range$(p+1)$ **do**
  Compute $m_1 = \left\langle {p \atop j} \right\rangle$ using Algorithm 14
  Compute $m_2 = \left(\frac{a}{d} + j + n - 1\right)_{p+1} - \left(\frac{a}{d} + j - 1\right)_{p+1}$
  $M = M + m_1 \times m_2$
end for
Compute $S_{p,a,d}(n) = \frac{d^p}{(p+1)!} \times M$
Return $S_{p,a,d}(n)$

In each part of this algorithm, the time complexity is $O(p)$ for the for-loop, $O(p^2)$ for classical Eulerian numbers, $O(p)$ for the Pochhammer symbol, and $O(p)$ for $\frac{d^p}{(p+1)!}$. So the time complexity overall is $O(p)(O(p^2) + O(p) + O(p)) = O(p^3)$.

3.5 Experiments

In this section, experimental results for evaluating the sums of powers of arithmetic progressions are presented. First, the specifications of the testing environment are given. Next, the details of the experiments are described. Lastly, the results of the original method versus the simplified method will be discussed.

3.5.1 Stirling Numbers of the Second Kind

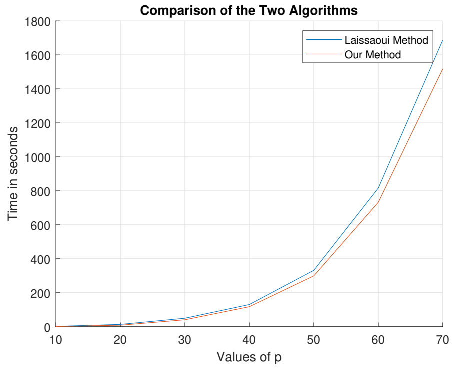
Specifications

All of the experiments are implemented using Python 3.7, with PyCharm IDE. The libraries includes Sympy and time. Sympy is a library that does symbolic mathematics. Data visualizations are done using Matlab. The experiments are run in a 64-bit Windows 10.0.18362 operating system. The computer used to run the tests consists of an Intel i7-8700K running at 3.7GHz with 6 physical cores and 16 GB of RAM.

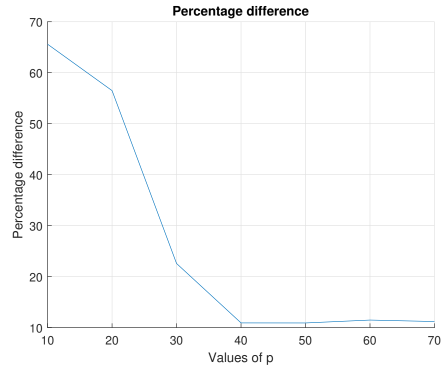
Description of Experiments

There are a total of seven experiments. In each experiment, the parameter p, which is the same p as in (3.13), varies. The first experiment sets $p = 10$. In subsequent experiments, p increments by 10 each time, with the last experiment testing on $p = 70$. The run-times are calculated using arithmetic mean over 100 trials.

In each experiment, the "stirling" function from the Sympy library is used to compute the Stirling numbers of the second kind. In addition, combinatorial functions such as "factorial" and "combsimp" are used. The functions for falling factorial and binomial are coded without dynamic programming.



(a) (a) A comparison of the run-times of the two algorithms shows that our method runs faster overall for all values of p.



(b) (b) The percentage difference in run-times of the two algorithms is consistent with our analysis.

Figure 3.1: Average run time of Laissaoui and Rahmani's method vs Our method.

Results and Discussion

The results of the experiments are shown in Figures (a) and (b). Figure (a) shows that the simplified version is superior to the original method from Theorem 1. Figure (b) shows the percentage difference between the two methods. For lower values of p, the percentage difference is much higher. Due to the p values being smaller, the overhead associated with the proposed algorithm may have contributed to the drastic difference. As the value of p increases, the simplified algorithm performs about 10% to 11% consistently better than the original.

3.5.2 Classical Eulerian Number

In this section, experimental results for evaluating the sums of powers of arithmetic progressions are presented. The specifications of the testing environment are given, and the details of the experiments are described. Thereafter, the results of the original method compared to the simplified method is be discussed.

Specifications

The computer used to run the tests consists of an Intel i7-8700K running at 3.7GHz with 6 physical cores and 16 gigabytes of RAM. The operating system used is Windows 10 Pro 20H2, 64 bit version. All of the experiments are implemented using Python 3.7 with Jupyter Notebook 6.0.3. The libraries include Sympy and time. Sympy is a library that does symbolic mathematics. The time library stores start-time and end-time, which are subtracted to obtain actual computation time. Mathematica is used to perform data visualizations.

Description of Experiments

In the first set of experiments, the value of the power p varies from $p = 10$ to $p = 100$ with $\Delta p = 10$ as the increments. This aims to give the sums of powers of arithmetic progressions in the general form. The results are simplified when comparing the results.

In the second set of experiments, the values of a, d, and n are fixed to randomly generated 20 digit numbers. The value of the power p varies from $p = 200$ to $p = 2000$ with $\Delta p = 200$ as the increments. This aims to better compare the two algorithms due to lower values of p may not give significant results. In both sets of experiments, the run-times are collected and averaged over 100 runs.

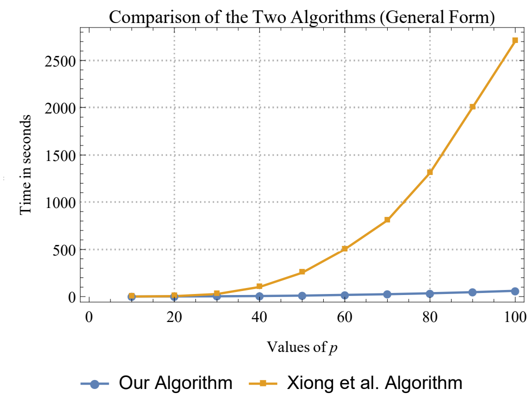
Results and Discussion

The results are visualized using Mathematica. The graphs are used to compare our algorithm (14) against Xiong *et al.*'s algorithm. Two tables are given to see the numerical results.

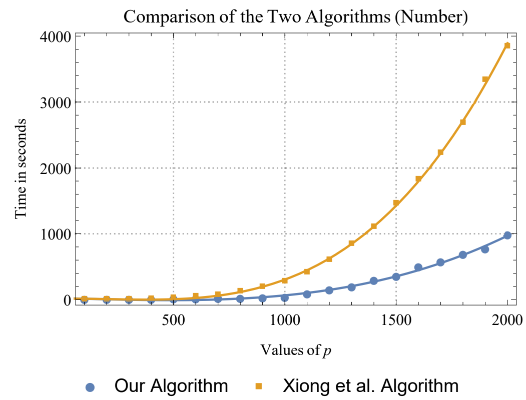
The main contribution to the drastic difference in times shown in figure (a) is that Xiong *et al.*'s algorithm gives the result in polynomials in n, whereas our algorithm gives the result in an unsimplified form. Even though the theoretical results can be simplified to the identity, a computer check was also performed to verify the answers are the same. This gives a more comparable result since the first set of experiments does not give the results in the same way.

Values of p	10	20	30	40	50
Our algorithm	0.4747	1.4272	3.1077	5.9152	10.2915
Xiong *et al.*'s algorithm	0.5635	4.5019	27.0317	101.6553	252.6017
Values of p	60	70	80	90	100
Our algorithm	17.8694	25.0377	34.7735	46.7552	61.1408
Xiong *et al.*'s's algorithm	500.5395	807.2444	1309.9709	2003.4787	2703.5214

Table 3.2: Experiment One: of The time is measured in seconds. Algorithms generate the general form of the sums of powers of arithmetic progressions. This table shows that our algorithm is far more efficient.



(a) Figure (a). A comparison of the two algorithms that calculates the general form of the sums of powers of arithmetic progressions by varying p only shows that our algorithm is drastically more efficient.



(b) Figure (b). A comparison of the two algorithms that calculate the result of the sums of powers of arithmetic progressions by fixing values of a, d, and n and varying p. The plot includes the line of best fit for two data sets.

Figure 3.2: Comparisons of two algorithms: ours vs Xiong et al. [97].

Values of p	100	200	300	400	500
Our algorithm	0.4497	1.0876	2.1052	3.9748	5.9838
Xiong *et al.*'s algorithm	0.359	1.3106	4.1687	10.9749	22.7876
Values of p	600	700	800	900	1000
Our algorithm	9.2827	13.244	17.6911	23.2921	34.1791
Xiong *et al.*'s algorithm	44.2563	70.7529	124.8384	188.3088	273.7294
Values of p	1100	1200	1300	1400	1500
Our algorithm	82.4342	145.502	188.3088	290.617	346.7947
Xiong *et al.*'s algorithm	412.5691	604.4786	846.1521	1105.5296	1457.421
Values of p	1600	1700	1800	1900	2000
Our algorithm	492.055	570.5803	681.9292	770.6358	979.0942
Xiong *et al.*'s algorithm	1822.8166	2221.2837	2681.4324	3336.1961	3840.6724

Table 3.3: Experiment Two: The time is measured in seconds. The constants a, d, and n are fixed to randomly generated 20 digit numbers, with the value of p varies. This table shows that our algorithm is faster in generating the values of sums of powers of arithmetic progressions.

3.6 Conclusion

In this chapter, we have discussed the theorems and the corresponding algorithms on computing sums of powers of arithmetic progressions using Stirling number of the second kind and classical Eulerian number.

For Stirling number of the second kind, we provided both an elementary proof and a simplification to Laissaoui and Rahmani's method for computing $S_{p,a,d}(n)$. The elementary proof given in Section 3.3.1 is an alternative proof to the one found in the original paper (Laissaoui and Rahmani [47]). Corollary 3.3.2 is a simplification to the original formula given in (3.13). It uses falling factorials instead of binomial coefficients.

For classical Eulerian number, we provided two novel algorithms to compute $S_{p,a,d}(n)$. We compared against the method of Xiong et al. [97], which used the general Eulerian numbers. We showed that our method with classical Eulerian number is more efficient compared with using general Eulerian number.

The publications can be seen in Shiue et al. [80, 79].

CHAPTER 4

COMPUTING RAMANUJAN CUBIC EQUATION AND RAMANAUJAN-TYPE IDENTITIES

4.1 Background

In this chapter, we considered properties of a theorem of Ramanujan to develop properties and algorithms related to cubic equations. Next, a generalized Computation procedure for construction of the Ramanujan-type from a given general cubic equation and a cosine Ramanujan-type identity is developed from detailed analyses of the properties of Ramanujan-type cubic equations. Examples are provided together with cubic Shevelev sums.

Ramanujan [68] wrote the following identities in his notebook :

$$\sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{8\pi}{7}}=\sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}}, \tag{4.1}$$

$$\sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=-\sqrt[3]{\frac{6-3\sqrt[3]{9}}{2}}, \tag{4.2}$$

$$\frac{1}{\sqrt[3]{\cos\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{\cos\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{\cos\frac{8\pi}{7}}}=\sqrt[3]{8-6\sqrt[3]{7}}, \tag{4.3}$$

$$\frac{1}{\sqrt[3]{\cos\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{\cos\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{\cos\frac{8\pi}{9}}}=-\sqrt[3]{6-6\sqrt[3]{9}} \tag{4.4}$$

Next, consider the following theorem of Ramanujan [68]:

Theorem 4.1.1. *Let α, β, and γ denote the distinct roots of a cubic equation*

$$x^3-ax^2+bx-1=0. \tag{4.5}$$

If α, β, and γ are real and distinct and the cubic roots of these numbers below are real, then

$$\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}=\sqrt[3]{a+6+3t} \quad \textit{and} \tag{4.6}$$

$$\frac{1}{\sqrt[3]{\alpha}}+\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}}=\sqrt[3]{b+6+3t}, \tag{4.7}$$

where t is the only real root of the associated Ramanujan equation

$$t^3-3(a+b+3)t-(ab+6(a+b)+9)=0 \tag{4.8}$$

with $t \neq \alpha,\ \beta,\ \gamma$.

Wang [90], inspired by Ramanujan's identities involving cosine functions, investigated similar identities involving sine and tangent functions. By using Ramanujan's theorem and Cardano's formula, he provided the following identities:

$$\sqrt[3]{\sin\frac{2\pi}{7}}+\sqrt[3]{\sin\frac{4\pi}{7}}+\sqrt[3]{\sin\frac{8\pi}{7}}=\left(-\sqrt[18]{\frac{7}{64}}\right)\left(\sqrt[3]{-\sqrt[3]{7}+6+3\left(\sqrt[3]{5-3\sqrt[3]{7}}+\sqrt[3]{4-3\sqrt[3]{7}}\right)}\right), \tag{4.9}$$

$$\frac{1}{\sqrt[3]{\sin\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{\sin\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{\sin\frac{8\pi}{7}}}=\left(-\sqrt[18]{\frac{64}{7}}\right)\left(\sqrt[3]{6+3\left(\sqrt[3]{5-3\sqrt[3]{7}}+\sqrt[3]{4-3\sqrt[3]{7}}\right)}\right), \tag{4.10}$$

$$\sqrt[3]{\sin\frac{\pi}{9}}+\sqrt[3]{\sin\frac{2\pi}{9}}+\sqrt[3]{\sin\frac{14\pi}{9}}=-\frac{\sqrt[18]{3}}{2}\sqrt[3]{6+3\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}, \tag{4.11}$$

$$\frac{1}{\sqrt[3]{\sin\frac{\pi}{9}}}+\frac{1}{\sqrt[3]{\sin\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{\sin\frac{14\pi}{9}}}=-\frac{2}{\sqrt[18]{3}}\sqrt[3]{-\sqrt[3]{9}+6+3\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}, \tag{4.12}$$

$$\begin{aligned}&\sqrt[3]{\tan\frac{2\pi}{7}}+\sqrt[3]{\tan\frac{4\pi}{7}}+\sqrt[3]{\tan\frac{8\pi}{7}}\\&\qquad=\left(\sqrt[18]{7}\right)\sqrt[3]{-\sqrt[3]{7}+6+3\left(\sqrt[3]{3\sqrt[3]{7}-3\sqrt[3]{49}+5}+\sqrt[3]{3\sqrt[3]{7}-3\sqrt[3]{49}-3}\right)},\end{aligned} \tag{4.13}$$

$$\frac{1}{\sqrt[3]{\tan\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{\tan\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{\tan\frac{8\pi}{7}}}$$

$$= \left(-\frac{1}{\sqrt[18]{7}}\right)\sqrt[3]{-\sqrt[3]{49}+6+3\left(\sqrt[3]{3\sqrt[3]{7}-3\sqrt[3]{49}+5}+\sqrt[3]{3\sqrt[3]{7}-3\sqrt[3]{49}-3}\right)}, \tag{4.14}$$

$$\sqrt[3]{\tan\frac{\pi}{9}}+\sqrt[3]{\tan\frac{4\pi}{9}}+\sqrt[3]{\tan\frac{7\pi}{9}}$$
$$= \left(-\sqrt[18]{3}\right)\left(\sqrt[3]{-3\sqrt[3]{3}+6+3\left(\sqrt[3]{21-3\left(3\sqrt[3]{3}+\sqrt[3]{9}\right)}-\sqrt[3]{3+3\left(3\sqrt[3]{3}+\sqrt[3]{9}\right)}\right)}\right), \tag{4.15}$$

$$\frac{1}{\sqrt[3]{\tan\frac{\pi}{9}}}+\frac{1}{\sqrt[3]{\tan\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{\tan\frac{7\pi}{9}}}$$
$$= \left(-\frac{1}{\sqrt[18]{3}}\right)\left(\sqrt[3]{-\sqrt[3]{9}+6+3\left(\sqrt[3]{21-3\left(3\sqrt[3]{3}+\sqrt[3]{9}\right)}-\sqrt[3]{3+3\left(3\sqrt[3]{3}+\sqrt[3]{9}\right)}\right)}\right). \tag{4.16}$$

In this chapter, we use the work of Liao et al. [50] and Chen [14] to determine the root t from the associated Ramanujan equation (4.8), with t being the only one real root; it is proved in Theorem 4.3.6. We apply this results to cubic equations of the various forms defined in Section 4.2. We also present a theorem to construct cosine Ramanujan-type identities. Lastly, we give the result for computing the Cubic Shevelev Sum (Shevelev [77]). In Section 4.6, we give a comprehensive computation procedure for cubic equations of categories Type 1, Type 2, and General Case. In Section 4.7, a great number of examples are presented.

4.2 Preliminaries

In this section, we present the preliminary information needed to support our main results. First, we classify the cubic equations into the following types:

(i). $x^3 - ax^2 + bx - 1 = 0,\ a + b + 3 \neq 0$, $a, b \in \mathbb{R}$, a, b not both 0 (Type 1)

(ii). $x^3 - ax^2 + bx - 1 = 0,\ a + b + 3 = 0$, $a, b \in \mathbb{R}$, a, b not both 0 (Type 2)

(iii). $x^3 + Ax^2 + Bx + C = 0,\ A\sqrt[3]{C} + B + 3\sqrt[3]{C^2} = 0$. (RCP 1)

(iv). $x^3 + Ax^2 + Bx + C = 0,\ B^3 + A^3C + 27C^2 = 0$. (RCP 2) (v). General case (none of the above)

Next, Shevelev [77] defined the sum,

$$\sqrt[3]{\frac{\alpha}{\beta}} + \sqrt[3]{\frac{\beta}{\gamma}} + \sqrt[3]{\frac{\gamma}{\alpha}} + \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{\gamma}{\beta}} + \sqrt[3]{\frac{\alpha}{\gamma}},$$

as the cubic Shevelev sum, where α, β, and γ are roots of a cubic equation $f(x) = x^3 + Ax^2 + Bx + C = 0$.

One may use Vieta's formula (Gilbert and Gilbert [24]) to verify the roots are of the cubic equations defined in the examples. In Wang's proofs for these identities, large amount of calculations are involved. Specifically, his computation is based on his Theorem 1.7, which relies on computing

$$\left(\left(\frac{\alpha}{\beta} + \frac{\beta}{\gamma} + \frac{\gamma}{\alpha}\right) - \left(\frac{\beta}{\alpha} + \frac{\gamma}{\beta} + \frac{\alpha}{\gamma}\right)\right)^2,$$

where α, β, and γ are the roots of $x^3 - ax^2 + bx - 1 = 0$.

The next theorem is used to prove our theorem to provide a more efficient method compared to Wang's approach.

Theorem 4.2.1. *(Liao et al. [50]). Given a polynomial equation of degree three* $x^3 + px + q = 0$ $(p, q \neq 0)$*, with real coefficients. Let* $p = -3rs$ *and* $q = rs(r + s)$*. Then the three solutions to this equation are*

$$x = -r^{\frac{1}{3}}s^{\frac{1}{3}}\left(r^{\frac{1}{3}} + s^{\frac{1}{3}}\right),\ -r^{\frac{1}{3}}s^{\frac{1}{3}}\left(\omega r^{\frac{1}{3}} + \omega^2 s^{\frac{1}{3}}\right),\ -r^{\frac{1}{3}}s^{\frac{1}{3}}\left(\omega^2 r^{\frac{1}{3}} + \omega s^{\frac{1}{3}}\right),$$

where $\omega = \frac{-1+\sqrt{3}i}{2}$.

Moreover, if $r, s \in \mathbb{R}$ *and* $r \neq s$*, then the equation has one real root and a pair of complex conjugate roots.*

4.3 Main Results

4.3.1 Type 1 & 2

Theorem 4.3.1. *(Shiue et al. [82]). Let $f(x) = x^3 - ax^2 + bx - 1 = 0$, $a, b \in \mathbb{R}$, a, b not both 0. Let $f(x)$ have three distinct real roots α, β, and γ. Let $t^3 - 3(a+b+3)t - (ab + 6(a+b) + 9) = 0$ be the associated Ramanujan equation. Then*

$$t = \sqrt[3]{\frac{ab + 6(a+b) + 9 + \Delta}{2}} + \sqrt[3]{\frac{ab + 6(a+b) + 9 - \Delta}{2}}, \tag{4.17}$$

where the discriminant of $f(x)$ is $\Delta^2 = (ab)^2 - 4(a^3 + b^3) + 18ab - 27$. Moreover,

$$\begin{aligned}\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} &= \sqrt[3]{a + 6 + 3t},\\ \frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} &= \sqrt[3]{b + 6 + 3t}.\end{aligned}$$

Proof. Using the method of Liao et al. [50], we have

$$-3rs = -3(a+b+3) \implies rs = a + b + 3, \tag{4.18}$$

$$rs(r+s) = -(ab + 6(a+b) + 9) \implies r + s = -\frac{(ab + 6(a+b) + 9)}{rs}. \tag{4.19}$$

To compute r and s, we compute $r - s$. Then

$$\begin{aligned}(r-s)^2 = (r+s)^2 - 4rs &= \frac{1}{(rs)^2}\left((ab + 6(a+b) + 9)^2 - 4(a+b+3)^3\right)\\ &= \frac{1}{(rs)^2}\Big[(ab)^2 + 12a^2b + 36a^2 + 12ab^2 + 90ab + 108a + 36b^2 + 108b + 81\\ &\quad - 4\left(a^3 + 3a^2b + 9a + 3ab^2 + 18ab + 27a + b^3 + 9b^2 + 27b + 27\right)\Big]\\ &= \frac{1}{(rs)^2}\left((ab)^2 - 4\left(a^3 + b^3\right) + 18ab - 27\right)\\ &\implies (r-s)^2(rs)^2 = (ab)^2 - 4\left(a^3 + b^3\right) + 18ab - 27.\end{aligned}$$

Let $\Delta^2 = (ab)^2 - 4\left(a^3 + b^3\right) + 18ab - 27 = (r-s)^2(rs)^2$. Note that the discriminant of $t^3 + pt + q = 0$

is always a perfect square. Then

$$r - s = \frac{\Delta}{rs}.$$

Hence,

$$\begin{aligned}2r &= -\frac{ab+6(a+b)+9}{rs} + \frac{\Delta}{rs} \implies r^2 s = \frac{-(ab+6(a+b)+9)+\Delta}{2},\\ 2s &= -\frac{ab+6(a+b)+9}{rs} - \frac{\Delta}{rs} \implies rs^2 = \frac{-(ab+6(a+b)+9)-\Delta}{2}.\end{aligned}$$

Thus,

$$\begin{aligned}t &= -(rs)^{\frac{1}{3}}\left(r^{\frac{1}{3}} + s^{\frac{1}{3}}\right) = -\left(r^{\frac{2}{3}}s^{\frac{1}{3}} + r^{\frac{1}{3}}s^{\frac{2}{3}}\right)\\ &= \sqrt[3]{\frac{(ab+6(a+b)+9)+\Delta}{2}} + \sqrt[3]{\frac{(ab+6(a+b)+9)-\Delta}{2}}.\end{aligned}$$

From Theorem 4.3.1, we have the second result. □

We apply this theorem to the Shanks polynomial (Dresden et al. [22]).

Corollary 4.3.2. (Shanks Polynomial Dresden et al. [22]). *Let* $f(x) = x^3 - ax^2 - (a+3)x - 1 = 0$. *The associated Ramanujan equation is* $t^3 + q = 0$, *where* $q = (a(a+3)+9)$. *Then* $t = -\sqrt[3]{a^2+3a+9}$. *The roots of* $f(x)$ *are*

$$\begin{aligned}&\frac{1}{3}\left(a + 2\sqrt{a^2+3a+9}\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)\right) \quad \textit{if} \quad a \geq -\frac{3}{2},\\ &\frac{1}{3}\left(a - 2\sqrt{a^2+3a+9}\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)\right) \quad \textit{if} \quad a \leq -\frac{3}{2},\end{aligned}$$

where $k = 0, 2, 4$. *Then by Theorem 4.3.1, we have for* $a \geq -\frac{3}{2}$, $k = 0, 2, 4$,

$$\begin{aligned}&\sum \sqrt[3]{\frac{1}{3}\left(a + 2\sqrt{a^2+3a+9}\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)\right)} = \sqrt[3]{a+6-3\sqrt[3]{a^2+3a+9}}\\ \implies &\sum \sqrt[3]{\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)} = \sqrt[3]{\frac{3}{a+2\sqrt{a^2+3a+9}}}\sqrt[3]{a+6-3\sqrt[3]{a^2+3a+9}}.\end{aligned}$$

For $a \leq -\dfrac{3}{2}$, $k = 0, 2, 4$, *we have*

$$\sum \sqrt[3]{\frac{1}{3}\left(a - 2\sqrt{a^2+3a+9}\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)\right)} = \sqrt[3]{a+6-3\sqrt[3]{a^2+3a+9}}$$

$$\Longrightarrow \sum \sqrt[3]{\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)} = \frac{\sqrt[3]{3}}{\sqrt[3]{a-2\sqrt{a^2+3a+9}}}\sqrt[3]{a+6-3\sqrt[3]{a^2+3a+9}}.$$

On the other hand, for $a \geq -\dfrac{3}{2}$, $k = 0, 2, 4$,

$$\sum \frac{1}{\sqrt[3]{\frac{1}{3}\left(a + 2\sqrt{a^2+3a+9}\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)\right)}} = \sqrt[3]{-(a+3)+6-3\sqrt[3]{a^2+3a+9}}$$

$$\Longrightarrow \sum \frac{1}{\sqrt[3]{\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)}} = \sqrt[3]{\frac{a+2\sqrt{a^2+3a+9}}{3}}\sqrt[3]{-a+3-3\sqrt[3]{a^2+3a+9}}.$$

For $a \leq -\dfrac{3}{2}$, $k = 0, 2, 4$, *we have*

$$\sum \frac{1}{\sqrt[3]{\frac{1}{3}\left(a - 2\sqrt{a^2+3a+9}\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)\right)}} = \sqrt[3]{-(a+3)+6-3\sqrt[3]{a^2+3a+9}}$$

$$\Longrightarrow \sum \frac{1}{\sqrt[3]{\cos\left(\frac{1}{3}\left(\arctan\left(\frac{3\sqrt{3}}{3+2a}\right)+k\pi\right)\right)}} = \sqrt[3]{\frac{a-2\sqrt{a^2+3a+9}}{3}}\sqrt[3]{-a+3-3\sqrt[3]{a^2+3a+9}}.$$

Corollary 4.3.3. *From Theorem 4.3.1, if* $a + b + 3 = 0$, *then* $t = \sqrt[3]{ab-9}$.

The condition $a+b+3=0$ was researched extensively by several authors Berndt and Bhargava [7], Dresden et al. [22], Shevelev [77]. Examples with this condition will be given in Section 4.7.

Theorem 4.3.4. *Let* $f(x) = x^3 - ax^2 + bx - 1 = 0$ *with* $a + b + 3 = 0$. *Then all three roots are distinct.*

Proof. From Theorem 4.3.1, the discriminant of $f(x) = 0$ is:

$$\Delta^2 = (ab)^2 - 4(a^3 + b^3) + 18ab - 27.$$

Then:

$$
\begin{aligned}
D &= a^2b^2 - 4b^3 - 4a^3 + 18ab - 27 = a^2b^2 - 4(a^3 + b^3) + 18ab - 27 \\
&= a^2b^2 - 4(a+b)(a^2 - ab + b^2) + 18ab - 27 = a^2b^2 + 12(a^2 - ab + b^2) + 18ab - 27 \\
&= a^2b^2 + 12(a^2 + b^2) + 6ab - 27 = a^2b^2 + 12[(a+b)^2 - 2ab] + 6ab - 27 \\
&= a^2b^2 + 12(9 - 2ab) + 6ab - 27 = a^2b^2 - 18ab + 81 = (ab - 9)^2 \geq 0.
\end{aligned}
$$

Therefore $f(x) = 0$ with $a + b + 3 = 0$ has three distinct roots if $ab \neq 9$. If $ab = 9$, then combined with $a + b + 3 = 0$, $b^2 + 3b + 9 = 0 \implies b$ is not real. □

Corollary 4.3.5. *Let $f(x) = x^3 - ax^2 + bx - 1$ with $a + b + 3 = 0$. The roots of $f(x)$ are the negatives of $g(x) = x^3 + ax^2 + bx + 1$ with $a + b + 3 = 0$. Then $g(x) = 0 = g\left(\frac{1}{1-x}\right), x \neq 1$.*

Proof.

$$
\begin{aligned}
g\left(\frac{1}{1-x}\right) = \left(\frac{1}{1-x}\right)^3 + a\left(\frac{1}{1-x}\right)^2 + b\left(\frac{1}{1-x}\right) + 1 \\
&= \frac{1}{(1-x)^3}[1 + a(1-x) + b(1-x)^2 + (1-x)^3] \\
&= \frac{1}{(1-x)^3}[1 + a - ax + b - 2bx + bx^2 + (1 - 3x + 3x^2 - x^3)] \\
&= \frac{1}{(1-x)^3}[-x^3 + (3+b)x^2 - (a + 2b + 3)x + (1 + a + b + 1)] \\
&= \frac{1}{(1-x)^3}[-x^3 - ax^2 - bx - 1] = 0 \\
&\implies x^3 + ax^2 + bx + 1 = 0 \\
&\implies g\left(\frac{1}{1-x}\right) = 0 = g(x).
\end{aligned}
$$

□

Theorem 4.3.6. *Let $t^3 + pt + q = 0$ be a cubic equation, where $p = -3(a + b + 3)$ and $q = -(ab + 6(a+b) + 9)$, $a, b \in \mathbb{R}$, a, b not both 0. Then only one root is real.*

Theorems 4.3.1 and 4.3.6 are used in the next section to prove the main theorems for the general case: $x^3 + Ax^2 + Bx + C = 0$, $A, B, C \in \mathbb{R}$, A, B not both 0, and $C \neq 0$.

4.3.2 General Case

Theorem 4.3.7. *Let $f(x) = x^3 + Ax^2 + Bx + C = 0$, $A, B, C \in \mathbb{R}$, A, B not both 0, $C \neq 0$. Let $f(x)$ have three distinct real roots α, β, and γ. Then*

(a) the associated Ramanujan equation is

$$t^3 - 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right)t - \left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = 0; \tag{4.20}$$

(b) the only real root to the associated Ramanujan equation (4.20) is

$$t = \sqrt[3]{\frac{\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 + \frac{\sqrt{D(f)}}{C}}{2}} + \sqrt[3]{\frac{\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 - \frac{\sqrt{D(f)}}{C}}{2}}, \tag{4.21}$$

where

$$D(f) = (AB)^2 - 4(A^3C + B^3) + 18ABC - 27C^2; \tag{4.22}$$

(c) the Ramanujan-type identities are

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = -\sqrt[3]{A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}}, \tag{4.23}$$

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = -\frac{1}{\sqrt[3]{C}}\sqrt[3]{B + 6\sqrt[3]{C^2} + 3t\sqrt[3]{C^2}}. \tag{4.24}$$

Proof. Let $f(x) = x^3 + Ax^2 + Bx + C = 0$, $C \neq 0$. Dividing by $-C$ yields

$$\left(-\frac{x}{\sqrt[3]{C}}\right)^3 + \left(-\frac{A}{\sqrt[3]{C}}\right)\left(-\frac{x}{\sqrt[3]{C}}\right)^2 + \left(\frac{B}{\sqrt[3]{C^2}}\right)\left(-\frac{x}{\sqrt[3]{C}}\right) - 1 = 0.$$

Let $y = -\frac{x}{\sqrt[3]{C}}$, $a = \frac{A}{\sqrt[3]{C}}$, and $b = \frac{B}{C^{\frac{2}{3}}}$. Then

$$g(y) = y^3 - ay^2 + by - 1 = 0. \tag{4.25}$$

By Ramanujan [68], the associated Ramanujan equation of (4.25) is $t^3 - 3(a + b + 3)t - (ab +$

$6(a+b)+9)=0$. Thus the associated Ramanujan equation of $x^3+Ax^2+Bx+C=0$ is

$$t^3-3\left(\frac{A}{\sqrt[3]{C}}+\frac{B}{\sqrt[3]{C^2}}+3\right)t-\left(\frac{AB}{C}+6\left(\frac{A}{\sqrt[3]{C}}+\frac{B}{\sqrt[3]{C^2}}\right)+9\right)=0, \tag{4.26}$$

which is the result for (a).

For (b), by Theorem 4.3.1,

$$\begin{aligned} D(g) &= (ab)^2-4(a^3+b^3)+18ab-27 \\ &= \left(\frac{AB}{C}\right)^2-4\left(\frac{A^3}{C}+\frac{B^3}{C^2}\right)+\frac{18AB}{C}-27 \\ &= \frac{1}{C^2}\left((AB)^2-4(A^3C+B^3)+18ABC-27C^2\right) \\ &= \frac{D(f)}{C^2}>0, \end{aligned}$$

since $f(x)=0$ has three distinct real roots.

By Theorem 4.3.6, the associated Ramanujan equation $t^3-3(a+b+3)t-(ab+6(a+b)+9)=0$ has only one real root t. Hence,

$$\begin{aligned} t &= \sqrt[3]{\frac{ab+6(a+b)+9+\sqrt{D(g)}}{2}}+\sqrt[3]{\frac{ab+6(a+b)+9-\sqrt{D(g)}}{2}} \\ &= \sqrt[3]{\frac{\frac{AB}{C}+6\left(\frac{A}{\sqrt[3]{C}}+\frac{B}{\sqrt[3]{C^2}}\right)+9+\frac{\sqrt{D(f)}}{C}}{2}}+\sqrt[3]{\frac{\frac{AB}{C}+6\left(\frac{A}{\sqrt[3]{C}}+\frac{B}{\sqrt[3]{C^2}}\right)+9-\frac{\sqrt{D(f)}}{C}}{2}} \end{aligned}$$

also has only one real root t. This concludes the proof for (b).

Lastly, for (c): note that the roots of (4.25) are $\alpha_y=-\frac{\alpha}{\sqrt[3]{C}}$, $\beta_y=-\frac{\beta}{\sqrt[3]{C}}$, and $\gamma_y=-\frac{\gamma}{\sqrt[3]{C}}$. Then, by Theorem 4.1.1, we have

$$\sqrt[3]{\alpha_y}+\sqrt[3]{\beta_y}+\sqrt[3]{\gamma_y}=\sqrt[3]{a+6+3t}, \tag{4.27}$$

$$\frac{1}{\sqrt[3]{\alpha_y}}+\frac{1}{\sqrt[3]{\beta_y}}+\frac{1}{\sqrt[3]{\gamma_y}}=\sqrt[3]{b+6+3t}, \tag{4.28}$$

where

$$t = \sqrt[3]{\frac{ab + 6(a+b) + 9 + \sqrt{D(g)}}{2}} + \sqrt[3]{\frac{ab + 6(a+b) + 9 - \sqrt{D(g)}}{2}}$$

and

$$D(g) = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = \frac{D(f)}{C^2}.$$

Substituting $\alpha_y = -\frac{\alpha}{\sqrt[3]{C}}$, $\beta_y = -\frac{\beta}{\sqrt[3]{C}}$, $\gamma_y = -\frac{\gamma}{\sqrt[3]{C}}$, $a = \frac{A}{\sqrt[3]{C}}$, and $b = \frac{B}{\sqrt[3]{C^2}}$, we have

$$\sqrt[3]{-\frac{\alpha}{\sqrt[3]{C}}} + \sqrt[3]{-\frac{\beta}{\sqrt[3]{C}}} + \sqrt[3]{-\frac{\gamma}{\sqrt[3]{C}}} = \sqrt[3]{\frac{A}{\sqrt[3]{C}} + 6 + 3t}, \tag{4.29}$$

$$\frac{1}{\sqrt[3]{-\frac{\alpha}{\sqrt[3]{C}}}} + \frac{1}{\sqrt[3]{-\frac{\beta}{\sqrt[3]{C}}}} + \frac{1}{\sqrt[3]{-\frac{\gamma}{\sqrt[3]{C}}}} = \sqrt[3]{\frac{B}{\sqrt[3]{C^2}} + 6 + 3t}, \tag{4.30}$$

where t and $D(f)$ are described in (4.21) and (4.22) respectively. Finally, simplification gives

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = -\sqrt[3]{A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}},$$
$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = -\frac{1}{\sqrt[3]{C}}\sqrt[3]{B + 6\sqrt[3]{C^2} + 3t\sqrt[3]{C^2}}.$$

□

Corollary 4.3.8. *Let $f(x) = x^3 - 3rsx + rs(r+s) = 0$ with real coefficients. Then*

(a) the associated Ramanujan equation is

$$t^3 + 9\left(\sqrt[3]{\frac{rs}{(r+s)^2}} - 1\right)t + 9\left(2\sqrt[3]{\frac{rs}{(r+s)^2}} - 1\right) = 0;$$

(b) the only real root to the associated Ramanujan equation is

$$t = \sqrt[3]{\frac{9\left(1 - 2\sqrt[3]{\frac{rs}{(r+s)^2}}\right) + \frac{\sqrt{D(f)}}{C}}{2}} + \sqrt[3]{\frac{9\left(1 - 2\sqrt[3]{\frac{rs}{(r+s)^2}}\right) - \frac{\sqrt{D(f)}}{C}}{2}}, \tag{4.31}$$

where

$$\frac{D(f)}{C^2} = -\frac{27(r-s)^2}{(r+s)^2}; \tag{4.32}$$

(c) the Ramanujan-type identities are

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = -\sqrt[3]{6\sqrt[3]{rs(r+s)} + 3t\sqrt[3]{rs(r+s)}}, \tag{4.33}$$

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{\frac{3}{r+s} - \frac{6}{\sqrt[3]{rs(r+s)}} - \frac{3t}{\sqrt[3]{rs(r+s)}}}. \tag{4.34}$$

Proof. Let $A = 0$, $B = -3rs$, and $C = rs(r+s)$ in the results of Theorem 4.3.7.

For part (a), we have

$$t^3 - 3\left(\frac{-3rs}{\sqrt[3]{rs(r+s)}} + 3\right)t - \left(6\left(\frac{-3rs}{\sqrt[3]{rs(r+s)}}\right) + 9\right) = 0.$$

Simplifying gives

$$t^3 + 9\left(\sqrt[3]{\frac{rs}{(r+s)^2}} - 1\right)t + 9\left(2\sqrt[3]{\frac{rs}{(r+s)^2}} - 1\right) = 0.$$

For part (b),

$$t = \sqrt[3]{\frac{9\left(1 - 2\sqrt[3]{\frac{rs}{(r+s)^2}}\right) + \frac{\sqrt{D(f)}}{C}}{2}} + \sqrt[3]{\frac{9\left(1 - 2\sqrt[3]{\frac{rs}{(r+s)^2}}\right) - \frac{\sqrt{D(f)}}{C}}{2}},$$

where

$$\begin{aligned}
\frac{D(f)}{C^2} &= \frac{1}{C^2}\left((AB)^2 - 4(A^3C + B^3) + 18ABC - 27C^2\right) \\
&= \frac{1}{(rs)^2(r+s)^2}\left(108(rs)^3 - 27(rs)^2(r+s)^2\right) = \frac{1}{(rs)^2(r+s)^2}\left(-27(rs)^2(r-s)^2\right) \\
&= -\frac{27(r-s)^2}{(r+s)^2}.
\end{aligned}$$

For part (c),

$$\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma} = -\sqrt[3]{A+6\sqrt[3]{C}+3t\sqrt[3]{C}}$$
$$= -\sqrt[3]{6\sqrt[3]{rs(r+s)}+3t\sqrt[3]{rs(r+s)}}$$

and

$$\frac{1}{\sqrt[3]{\alpha}}+\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}} = -\frac{1}{\sqrt[3]{C}}\sqrt[3]{B+6\sqrt[3]{C^2}+3t\sqrt[3]{C^2}}$$
$$= -\frac{1}{\sqrt[3]{rs(r+s)}}\sqrt[3]{-3rs+6\sqrt[3]{(rs)^2(r+s)^2}+3t\sqrt[3]{(rs)^2(r+s)^2}}$$
$$= \sqrt[3]{\frac{3}{r+s}-\frac{6}{\sqrt[3]{rs(r+s)}}-\frac{3t}{\sqrt[3]{rs(r+s)}}}.$$

□

Next, we shall use Theorems 4.2.1 and 4.3.7 to obtain Theorem 4.3.9 to construct cosine Ramanujan-type identities. An example of the construction is given in section 4.7.6.

Theorem 4.3.9. *Let $f(x) = x^3 - 3rsx + rs(r+s) = 0$, where $r, s \in \mathbb{C}$ are complex conjugates with $r = \xi + \eta i$ and $\xi, \eta \neq 0$. Define $\theta = \operatorname{Arg}(r)$. Then the cosine Ramanujan-type identities can be obtained via the following:*

$$\sqrt[3]{\cos\left(\frac{\theta}{3}\right)}+\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}+\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)} = \sqrt[9]{\frac{2\xi}{|r|}}\sqrt[3]{3+\frac{3t}{2}}, \tag{4.35}$$

$$\frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}\right)}}+\frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}}+\frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}} = \sqrt[3]{\frac{3}{\xi}}\sqrt[3]{-|r|+2\sqrt[3]{4\xi^2|r|}+t\sqrt[3]{4\xi^2|r|}}, \tag{4.36}$$

where

$$t = \sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2|r|}{\xi^2}}\right)+3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}+\sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2|r|}{\xi^2}}\right)-3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}. \tag{4.37}$$

Proof. By Theorem 4.2.1, the three distinct roots of $f(x) = 0$ are

$$\alpha = -2\sqrt{rs}\cos\left(\frac{\theta}{3}\right), \beta = -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right), \gamma = -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right).$$

Note that $rs = |r|^2$ and $r + s = 2\xi$. Using (4.26), the associated Ramanujan equation is

$$\begin{aligned} 0 &= t^3 - 3\left(\frac{-3rs}{\sqrt[3]{(rs)^2(r+s)^2}} + 3\right)t - 6\left(\frac{-3rs}{\sqrt[3]{(rs)^2(r+s)^2}}\right) - 9 \\ &= t^3 + 9\left(\sqrt[3]{\frac{rs}{(r+s)^2}} - 1\right)t - 9\left(-2\sqrt[3]{\frac{rs}{(r+s)^2}} + 1\right) = 0 \\ &= t^3 + 9\left(\sqrt[3]{\frac{|r|}{4\xi^2}} - 1\right)t + 9\left(\sqrt[3]{\frac{2|r|}{\xi^2}} - 1\right). \end{aligned}$$

Using (4.22), the discriminant of $f(x) = 0$ is

$$\begin{aligned} \frac{D(f)}{C^2} &= \frac{1}{(rs)^2(r+s)^2}\left(108(rs)^3 - 27(rs)^2(r+s)^2\right) \\ &= \frac{1}{4\xi^2|r|^4}\left(108|r|^6 - 108\xi^2|r|^4\right) \\ &= \frac{27}{\xi^2}\left(|r|^2 - \xi^2\right) = \frac{27\eta^2}{\xi^2}. \end{aligned}$$

Next, taking the square root allows us to substitute into (4.21). Hence,

$$\frac{\sqrt{D(f)}}{C} = 3\sqrt{3}\left(\frac{\eta}{\xi}\right). \tag{4.38}$$

By (4.21) in Theorem 4.3.7, the real root t is

$$t = \sqrt[3]{\frac{9\left(1 - \sqrt[3]{\frac{2|r|}{\xi^2}}\right) + 3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}} + \sqrt[3]{\frac{9\left(1 - \sqrt[3]{\frac{2|r|}{\xi^2}}\right) - 3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}.$$

Then, by Theorem 4.3.7,

$$\begin{aligned} &\sqrt[3]{-2|r|\cos\left(\frac{\theta}{3}\right)} + \sqrt[3]{-2|r|\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)} + \sqrt[3]{-2|r|\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right)} \\ &\qquad = -\sqrt[9]{2\xi|r|^2}\,\sqrt[3]{6 + 3t}. \end{aligned}$$

Thus,

$$\sqrt[3]{\cos\left(\frac{\theta}{3}\right)} + \sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)} + \sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)} = \sqrt[9]{\frac{2\xi}{|r|}}\sqrt[3]{3+\frac{3t}{2}}.$$

On the other hand,

$$\frac{1}{\sqrt[3]{-2|r|\cos\left(\frac{\theta}{3}\right)}} + \frac{1}{\sqrt[3]{-2|r|\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}} + \frac{1}{\sqrt[3]{-2|r|\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}}$$
$$= -\frac{1}{\sqrt[3]{2\xi|r|^2}}\sqrt[3]{-3|r|^2 + 6\sqrt[3]{4|r|^4\xi^2} + 3t\sqrt[3]{4|r|^4\xi^2}}.$$

Thus,

$$\frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}\right)}} + \frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}} + \frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}} = \sqrt[3]{\frac{3}{\xi}}\sqrt[3]{-|r| + 2\sqrt[3]{4\xi^2|r|} + t\sqrt[3]{4\xi^2|r|}}.$$

□

Lemma 4.3.1. *Let $f(x) = x^3 + Ax^2 + Bx + C = 0$, where $A, B, C \in \mathbb{R}$, $A^2 \neq 3B$, and $C \neq 0$. Let its depressed cubic form be $x^3 - 3rsx + rs(r+s) = 0$. Denote $D(f)$ as the discriminant of $f(x)$.*

(a) If $D(f) > 0$, then r and s are complex conjugates.

(b) If $D(f) = 0$, then $r = s$ is a real number.

(c) If $D(f) < 0$, then r and s are distinct real numbers.

Proof. By substituting $x = y - \frac{A}{3}$ into $f(x)$, we get

$$\left(y - \frac{A}{3}\right)^3 + A\left(y - \frac{A}{3}\right)^2 + B\left(y - \frac{A}{3}\right) + C = 0,$$

which simplifies to the depressed cubic

$$y^3 - \frac{1}{3}\left(A^2 - 3B\right)y + \frac{1}{27}\left(2A^3 - 9AB + 27C\right) = 0. \tag{4.39}$$

Using the result from Theorem 4.2.1, we have

$$
\begin{aligned}
-3rs &= -\frac{1}{3}(A^2 - 3B), \\
rs(r+s) &= \frac{1}{27}\left(2A^3 - 9AB + 27C\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
rs &= \frac{A^2 - 3B}{9}, \\
r + s &= \frac{\left(2A^3 - 9AB + 27C\right)}{3(A^2 - 3B)}.
\end{aligned}
$$

Then r and s are roots of the quadratic equation $h(z) = z^2 - \frac{\left(2A^3 - 9AB + 27C\right)}{3(A^2 - 3B)} z + \frac{A^2 - 3B}{9}$. The discriminant of this quadratic equation is

$$
\begin{aligned}
D(h) &= \frac{\left(2A^3 - 9AB + 27C\right)^2}{9(A^2 - 3B)^2} - \frac{4(A^2 - 3B)}{9} \\
&= \frac{1}{9(A^2 - 3B)^2}\left(\left(2A^3 - 9AB + 27C\right)^2 - 4(A^2 - 3B)^3\right) \\
&= \frac{1}{9(A^2 - 3B)^2}\left(4A^6 - 36A^4B + 108A^3C + 81A^2B^2 - 486ABC + 729C^2\right) \\
&\quad - \frac{4}{9(A^2 - 3B)^2}\left(A^6 - 9A^4B + 27A^2B^2 - 27B^3\right) \\
&= -\frac{27}{9(A^2 - 3B)^2}\left(A^2B^2 - 4(A^3C + B^3) + 18ABC - 27C^2\right) = -\frac{3D(f)}{(A^2 - 3B)^2},
\end{aligned}
$$

where $D(f)$ is the discriminant of $f(x)$. Therefore, if $D(f) > 0$, then $D(h) < 0$. Hence, if the discriminant of $f(x)$ is positive, i.e., it has three distinct real roots, then r and s are complex conjugates.

Similarly, if $D(f) = 0$, then $D(h) = 0$. Hence, if the discriminant of $f(x)$ is zero, then $r = s$ is a real number. If $D(f) < 0$, then $D(h) > 0$. Hence, if the discriminant of $f(x)$ is negative, then r and s are distinct real numbers. □

Remark 4.3.1. *If $A^2 = 3B$, then (4.39) becomes $y^3 - \frac{A^3}{27} + C = 0$. The roots of this cubic equation are $\alpha_y = \frac{1}{3}\sqrt[3]{A^3 - 27C}$, $\beta_y = \omega\alpha_y$, and $\gamma_y = \omega^2\alpha_y$, where $\omega = \frac{-1+\sqrt{3}}{2}$. Hence, the roots for $f(x)$*

are $\alpha = \frac{1}{3}\sqrt[3]{A^3 - 27C} - \frac{A}{3}$, $\beta = \frac{\omega}{3}\sqrt[3]{A^3 - 27C} - \frac{A}{3}$, *and* $\gamma = \frac{\omega^2}{3}\sqrt[3]{A^3 - 27C} - \frac{A}{3}$.

Given a general cubic equation, Lemma 4.3.1 gives the condition of when r and$s$ are complex conjugates. Corollary 4.3.10 uses Theorem 4.3.9 to construct cosine Ramanujan-type identity given a particular form of cubic equation.

Corollary 4.3.10. *Let* $f(x) = x^3 + Ax^2 + \frac{A^2-9}{3}x + \frac{A^3}{27} - A + 2C$, *where* $A \in \mathbb{R}$ *and* $C \in (-1, 1)$. *Then the discriminant of* $f(x)$, $D(f)$, *is greater than 0. Moreover, let the distinct roots of* $f(x) = 0$ *be* α, β, *and* γ. *Then the cosine Ramanujan-type identities are*

$$\sqrt[3]{\alpha + \frac{A}{3}} + \sqrt[3]{\beta + \frac{A}{3}} + \sqrt[3]{\gamma + \frac{A}{3}} = -\sqrt[3]{6\sqrt[3]{2C} + 3t\sqrt[3]{2C}}, \tag{4.40}$$

and

$$\frac{1}{\sqrt[3]{\alpha + \frac{A}{3}}} + \frac{1}{\sqrt[3]{\beta + \frac{A}{3}}} + \frac{1}{\sqrt[3]{\gamma + \frac{A}{3}}} = -\frac{1}{\sqrt[3]{2C}}\sqrt[3]{-3 + 6\sqrt[3]{4C^2} + 3t\sqrt[3]{4C^2}}, \tag{4.41}$$

where

$$t = \sqrt[3]{\frac{9\left(1 - \sqrt[3]{\frac{2}{C^2}}\right) + 3\sqrt{3}\left(\frac{\sqrt{1-C^2}}{C}\right)}{2}} + \sqrt[3]{\frac{9\left(1 - \sqrt[3]{\frac{2}{C^2}}\right) - 3\sqrt{3}\left(\frac{\sqrt{1-C^2}}{C}\right)}{2}}. \tag{4.42}$$

Proof. Let $f(x) = x^3 + Ax^2 + \frac{A^2-9}{3}x + \frac{A^3}{27} - A + 2C = 0$, with $A \in \mathbb{R}$ and $C \in (-1, 1)$. The discriminant $D(f)$ is

$$\begin{aligned} D(f) = A^2\left(\frac{A^2-9}{3}\right) - 4\left(A^3\left(\frac{A^3}{27} - A + 2C\right) + \left(\frac{A^2-9}{3}\right)^3\right) \\ + 18A\left(\frac{A^2-9}{3}\right)\left(\frac{A^3}{27} - A + 2C\right) - 27\left(\frac{A^3}{27} - A + 2C\right)^2 \\ = 108(1 - C^2), \end{aligned}$$

which is greater than zero for $C \in (-1, 1)$.

Next, by substituting $x = y - \frac{A}{3}$ into $f(x)$, we get

$$\left(y - \frac{A}{3}\right)^3 + A\left(y - \frac{A}{3}\right)^2 + \left(\frac{A^2 - 9}{3}\right)\left(y - \frac{A}{3}\right) + \frac{A^3}{27} - A + 2C = 0,$$

which simplifies to the depressed cubic

$$y^3 - 3y + 2C = 0. \tag{4.43}$$

Denote the roots of (4.43) as α_y, β_y, and γ_y. Next, using Theorem 4.3.9, we have

$$\begin{aligned} -3rs &= -3, \\ rs(r+s) &= 2C. \end{aligned}$$

Then

$$\begin{aligned} rs &= 1, \\ r + s &= 2C. \end{aligned}$$

Then r, s are roots of the quadratic equation

$$z^2 - 2Cz + 1 = 0.$$

Using the quadratic formula,

$$\begin{aligned} r &= C + \sqrt{C^2 - 1} = C + i\sqrt{1 - C^2}, \\ s &= C - i\sqrt{1 - C^2}. \end{aligned}$$

To find the terms inside (4.37) in Theorem 4.3.9, we first find

$$\frac{\sqrt{D(f)}}{C} = 3\sqrt{3}\left(\frac{\sqrt{1 - C^2}}{C}\right).$$

Then

$$t=\sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{C^2}}\right)+3\sqrt{3}\left(\frac{\sqrt{1-C^2}}{C}\right)}{2}}+\sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{C^2}}\right)-3\sqrt{3}\left(\frac{\sqrt{1-C^2}}{C}\right)}{2}}.$$

By the result of Theorem 4.3.7, we have

$$\sqrt[3]{\alpha_y}+\sqrt[3]{\beta_y}+\sqrt[3]{\gamma_y}=-\sqrt[3]{6\sqrt[3]{2C}+3t\sqrt[3]{2C}}$$

and

$$\frac{1}{\sqrt[3]{\alpha_y}}+\frac{1}{\sqrt[3]{\beta_y}}+\frac{1}{\sqrt[3]{\gamma_y}}=-\sqrt[3]{-3+6\sqrt[3]{4C^2}+3t\sqrt[3]{4C^2}}.$$

Finally,

$$\sqrt[3]{\alpha+\frac{A}{3}}+\sqrt[3]{\beta+\frac{A}{3}}+\sqrt[3]{\gamma+\frac{A}{3}}=-\sqrt[3]{6\sqrt[3]{2C}+3t\sqrt[3]{2C}}$$

and

$$\frac{1}{\sqrt[3]{\alpha+\frac{A}{3}}}+\frac{1}{\sqrt[3]{\beta+\frac{A}{3}}}+\frac{1}{\sqrt[3]{\gamma+\frac{A}{3}}}=-\frac{1}{\sqrt[3]{2C}}\sqrt[3]{-3+6\sqrt[3]{4C^2}+3t\sqrt[3]{4C^2}}.$$

□

4.4 Ramanujan Cubic Polynomials

4.4.1 RCP

Shevelev [77] defined the Ramanujan cubic polynomials (RCP) as follows: Let $A, B, C \in \mathbb{R},\ C \neq 0$. The cubic polynomial $f(x) = x^3 + Ax^2 + Bx + C$ is RCP if it has three real roots and satisfies the condition $A\sqrt[3]{C} + B + 3\sqrt[3]{C^2} = 0$.

For the case of RCP, $f(x) = x^3 + Ax^2 + Bx + C = 0$ has three distinct roots (Shevelev [77]). We have the following theorem:

Theorem 4.4.1. *Let $f(x) = x^3 + Ax^2 + Bx + C = 0,\ A, B, C \in \mathbb{R},\ C \neq 0$, be an RCP. The root t*

of its associated Ramanujan equation, $t^3 - \left(\frac{AB}{C} - 9\right) = 0$, *is*

$$t = \sqrt[3]{\frac{AB}{C} - 9}, \tag{4.44}$$

and

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{-A - 6\sqrt[3]{C} + 3\sqrt[3]{9C - AB}}, \tag{4.45}$$

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \frac{1}{\sqrt[3]{C}}\sqrt[3]{-B - 6\sqrt[3]{C^2} + 3\sqrt[3]{9C^2 - ABC}}. \tag{4.46}$$

Proof. Let $f(x) = x^3 + Ax^2 + Bx + C = 0$, $A, B, C \in \mathbb{R}$, $C \neq 0$, and $A\sqrt[3]{C} + B + 3\sqrt[3]{C^2} = 0$. From Theorem 4.3.7, we have

$$\begin{aligned}
D(f) &= A^2B^2 - 4\left(\left(A\sqrt[3]{C}\right)^3 + B^3\right) + 18ABC - 27C^2 \\
&= A^2B^2 - 4\left(A\sqrt[3]{C} + B\right)\left(A^2\sqrt[3]{C^2} - AB\sqrt[3]{C} + B^2\right) + 18ABC - 27C^2 \\
&= A^2B^2 - 4\left(-3\sqrt[3]{C^2}\right)\left(A^2\sqrt[3]{C^2} - AB\sqrt[3]{C} + B^2\right) + 18ABC - 27C^2 \\
&= A^2B^2 + 12\sqrt[3]{C^2}\left(A^2\sqrt[3]{C^2} + B^2\right) + 6ABC - 27C^2 \\
&= A^2B^2 + 12\sqrt[3]{C^2}\left(\left(A\sqrt[3]{C} + B\right)^2 - 2AB\sqrt[3]{C}\right) + 6ABC - 27C^2 \\
&= A^2B^2 + 12\sqrt[3]{C^2}\left(9C^{\frac{4}{3}} - 2AB\sqrt[3]{C}\right) + 6ABC - 27C^2 \\
&= (AB)^2 + 81C^2 - 18ABC = (AB - 9C)^2 \geq 0.
\end{aligned}$$

From the definition of RCP, $A\sqrt[3]{C} + B + 3\sqrt[3]{C^2} = 0$, we have

$$\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3 = 0.$$

Therefore, the associated Ramanujan equation of $f(x)$ is

$$t^3 - \left(\frac{AB}{C} - 9\right) = 0.$$

Hence, the real root t to the associated Ramanujan equation is

$$t = \sqrt[3]{\frac{AB}{C} - 9}.$$

By Theorem 4.3.7,

$$\begin{aligned}\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} &= -\sqrt[3]{A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}} \\ &= \sqrt[3]{-A - 6\sqrt[3]{C} + 3\sqrt[3]{9C - AB}},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} &= -\frac{1}{\sqrt[3]{C}}\sqrt[3]{B + 6\sqrt[3]{C^2} + 3t\sqrt[3]{C^2}} \\ &= \frac{1}{\sqrt[3]{C}}\sqrt[3]{-B - 6\sqrt[3]{C^2} + 3\sqrt[3]{9C^2 - ABC}}.\end{aligned}$$

□

Remark 4.4.1. *Theorem 4.4.1 was proved by Shevelev [77] with a different proof.*

Dresden et al. [22] investigated the following cubic equation:

Corollary 4.4.2. *Let* $f(x) = x^3 - \frac{3+B}{2}x^2 - \frac{3-B}{2}x + 1 = 0$. *Then* $f(x)$ *is an RCP. Moreover, the Ramanujan-type identities are*

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{\frac{3+B}{2} - 6 + 3\sqrt[3]{\frac{27+B^2}{4}}}.$$

and

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{\frac{3-B}{2} - 6 + 3\sqrt[3]{\frac{27+B^2}{4}}}.$$

Proof. Substituting $A = -\frac{3+B}{2}$, $B = -\frac{3-B}{2}$, and $C = 1$ in the result of Theorem 4.4.1 yields the result of this corollary. □

Remark 4.4.2. *The results are proved differently in their paper.*

4.4.2 RCP2

Similarly, Wituła [95] defined the cubic polynomials of the form $f(x) = x^3 + Ax^2 + Bx + C$ with real roots and satisfying $B^3 + A^3C + 27C^2 = 0$ as Ramanujan cubic polynomials of the second kind (RCP2). For the case of RCP2, we have the following result.

Theorem 4.4.3. *Let $f(x) = x^3 + Ax^2 + Bx + C = 0$, $A, B, C \in \mathbb{R}$, $C \neq 0$, be an RCP2. Then the root t of the associated Ramanujan equation,*

$$t^3 - 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right)t - \left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = 0,$$

is

$$t = \sqrt[3]{\frac{1}{C}\left(A + 3\sqrt[3]{C}\right)\left(B + 3\sqrt[3]{C^2}\right)} + \sqrt[3]{\frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C} + B\right)}, \tag{4.47}$$

and

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{-A - 6\sqrt[3]{C} - 3\left(\sqrt[3]{\left(A + 3\sqrt[3]{C}\right)\left(B + 3\sqrt[3]{C^2}\right)} + \sqrt[3]{3\sqrt[3]{C}\left(A\sqrt[3]{C} + B\right)}\right)}, \tag{4.48}$$

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{-\frac{B}{C} - \frac{6}{\sqrt[3]{C}} - \frac{3}{\sqrt[3]{C^2}}\left(\sqrt[3]{\left(A + 3\sqrt[3]{C}\right)\left(B + 3\sqrt[3]{C^2}\right)} + \sqrt[3]{3\sqrt[3]{C}\left(A\sqrt[3]{C} + B\right)}\right)}. \tag{4.49}$$

Proof. From the condition of RCP2, we have

$$B^3 + A^3C + 27C^2 = 0.$$

From Theorem 4.3.7, we have

$$\begin{aligned} D(f) &= A^2B^2 - 4B^3 - 4A^3C + 18ABC - 27C^2 \\ &= A^2B^2 + 4C(A^3 + 27C) - 4A^3C + 18ABC - 27C^2 \end{aligned}$$

$$= A^2B^2 + 18ABC + 81C^2 = (AB + 9C)^2 \geq 0.$$

Then

$$\frac{D(f)}{C^2} = \left(\frac{AB}{C} + 9\right)^2.$$

Taking the square root allows us to substitute into (4.21):

$$\frac{\sqrt{D(f)}}{C} = \frac{AB}{C} + 9. \tag{4.50}$$

The associated Ramanujan equation of $f(x)$ is

$$t^3 - 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right)t - \left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = 0.$$

From (4.26), the real root t to the associated Ramanujan equation is

$$t = \sqrt[3]{\frac{\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 + \frac{\sqrt{D(f)}}{C}}{2}} + \sqrt[3]{\frac{\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 - \frac{\sqrt{D(f)}}{C}}{2}}.$$

Computing the quantities under the cube roots give:

$$\begin{aligned}
\frac{1}{2}\left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 + \frac{AB}{C} + 9\right) &= \frac{AB}{C} + 9 + 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) \\
&= \frac{1}{C}\left(AB + 9C + \left(A\sqrt[3]{C^2} + B\sqrt[3]{C}\right)\right) \\
&= \frac{1}{C}\left(A + 3\sqrt[3]{C}\right)\left(B + 3\sqrt[3]{C^2}\right)
\end{aligned}$$

and

$$\frac{1}{2}\left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 - \left(\frac{AB}{C} + 9\right)\right) = \frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C} + B\right).$$

Denote the roots of $f(x)$ by α, β, and γ. By Theorem 4.3.7, we have

$$\begin{aligned}\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma} &= -\sqrt[3]{A+6\sqrt[3]{C}+3t\sqrt[3]{C}} \\ &= \sqrt[3]{-A-6\sqrt[3]{C}-3\left(\sqrt[3]{\left(A+3\sqrt[3]{C}\right)\left(B+3\sqrt[3]{C^2}\right)}+\sqrt[3]{3\sqrt[3]{C}\left(A\sqrt[3]{C}+B\right)}\right)},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\sqrt[3]{\alpha}}+\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}} &= -\frac{1}{\sqrt[3]{C}}\sqrt[3]{B+6\sqrt[3]{C^2}+3t\sqrt[3]{C^2}} \\ &= \sqrt[3]{-\frac{B}{C}-\frac{6}{\sqrt[3]{C}}-\frac{3}{\sqrt[3]{C^2}}\left(\sqrt[3]{\left(A+3\sqrt[3]{C}\right)\left(B+3\sqrt[3]{C^2}\right)}+\sqrt[3]{3\sqrt[3]{C}\left(A\sqrt[3]{C}+B\right)}\right)}.\end{aligned}$$

□

Remark 4.4.3. *The first identity was proved by Wituła [95] but not the second identity.*

4.5 Cubic Shevelev Sum

Shevelev [77] defined the sum,

$$\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}},$$

as the cubic Shevelev sum, where α, β, and γ are roots of a cubic equation $f(x)=x^3+Ax^2+Bx+C=0$.

Remark 4.5.1. *Wang [90] investigated the following quantity similar to the cubic Shevelev sum by showing that the discriminant of $f(x)=x^3-ax^2+bx-1=0$, $D(f)$, is*

$$\left(\left(\frac{\alpha}{\beta}+\frac{\beta}{\gamma}+\frac{\gamma}{\alpha}\right)-\left(\frac{\beta}{\alpha}+\frac{\gamma}{\beta}+\frac{\alpha}{\gamma}\right)\right)^2,$$

where α, β, and γ, with each not equal to zero, are roots of $f(x)=0$.

Using Theorem 4.3.7, we have the following generalized cubic Shevelev sum.

Theorem 4.5.1. *Let $f(x) = x^3 + Ax^2 + Bx + C = 0$, $A, B, C \in \mathbb{R}$, A, B not both 0, $C \neq 0$. Let $f(x)$ have three distinct real roots α, β, and γ, with $\alpha,\ \beta,\ \gamma \neq 0$. Then*

$$\sqrt[3]{\frac{\alpha}{\beta}} + \sqrt[3]{\frac{\beta}{\gamma}} + \sqrt[3]{\frac{\gamma}{\alpha}} + \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{\gamma}{\beta}} + \sqrt[3]{\frac{\alpha}{\gamma}} = t, \tag{4.51}$$

where t is shown in (4.21).

Proof. First note that

$$\begin{aligned}\left(\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma}\right)&\left(\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}}\right)\\ &= \sqrt[3]{\frac{\alpha}{\beta}} + \sqrt[3]{\frac{\beta}{\gamma}} + \sqrt[3]{\frac{\gamma}{\alpha}} + \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{\gamma}{\beta}} + \sqrt[3]{\frac{\alpha}{\gamma}} + 3.\end{aligned} \tag{4.52}$$

On the other hand, using (4.23) and (4.24), we have

$$\begin{aligned}\left(\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma}\right)&\left(\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}}\right)\\ &= \left(-\sqrt[3]{A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}}\right)\left(-\sqrt[3]{\frac{B}{C} + \frac{6}{\sqrt[3]{C}} + \frac{3}{\sqrt[3]{C}}t}\right)\\ &= \sqrt[3]{\left(A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}\right)\left(\frac{B}{C} + \frac{6}{\sqrt[3]{C}} + \frac{3}{\sqrt[3]{C}}t\right)}.\end{aligned}$$

Simplifying the expression under the cube root, we have

$$\begin{aligned}&\left(A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}\right)\left(\frac{B}{C} + \frac{6}{\sqrt[3]{C}} + \frac{3}{\sqrt[3]{C}}t\right) = \left(\frac{A}{\sqrt[3]{C}} + 6 + 3t\right)\left(\frac{B}{\sqrt[3]{C^2}} + 6 + 3t\right)\\ &= \frac{AB}{C} + 6\frac{A}{\sqrt[3]{C}} + 3\frac{A}{\sqrt[3]{C}}t + 6\frac{B}{\sqrt[3]{C^2}} + 36 + 18t + 3\frac{B}{\sqrt[3]{C^2}}t + 18t + 9t^2\\ &= \frac{AB}{C} + \frac{6}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C} + B\right) + 36 + 36t + \frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C} + B\right)t + 9t^2.\end{aligned}$$

Recall the associated Ramanujan equation $t^3 + pt + q = 0$, where $p = -3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right)$ and $q = -\left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right)$. Then we have

$$\left(A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}\right)\left(\frac{B}{C} + \frac{6}{\sqrt[3]{C}} + \frac{3}{\sqrt[3]{C}}t\right)$$

$$= -q + 27 - pt + 27t + 9t^2 = t^3 + 9t^2 + 27t + 27 = (t+3)^3.$$

Hence,

$$\left(\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma}\right)\left(\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}}\right) = \sqrt[3]{(t+3)^3} = t + 3. \tag{4.53}$$

Comparing (4.52) and (4.53) gives

$$\sqrt[3]{\frac{\alpha}{\beta}} + \sqrt[3]{\frac{\beta}{\gamma}} + \sqrt[3]{\frac{\gamma}{\alpha}} + \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{\gamma}{\beta}} + \sqrt[3]{\frac{\alpha}{\gamma}} = t.$$

□

Applying Theorem 4.5.1, we have the following three corollaries.

Corollary 4.5.2. *(Shevelev [77]). If $f(x)$ is an RCP, then the cubic Shevelev sum is*

$$\sqrt[3]{\frac{\alpha}{\beta}} + \sqrt[3]{\frac{\beta}{\gamma}} + \sqrt[3]{\frac{\gamma}{\alpha}} + \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{\gamma}{\beta}} + \sqrt[3]{\frac{\alpha}{\gamma}} = \sqrt[3]{\frac{AB}{C} - 9}. \tag{4.54}$$

Proof. Let $f(x) = x^3 + Ax^2 + Bx + C = 0$ be a cubic equation satisfying Theorem 4.4.1. Then $t = \sqrt[3]{\frac{AB}{C} - 9}$. From Theorem 4.5.1, the cubic Shevelev sum is

$$\sqrt[3]{\frac{\alpha}{\beta}} + \sqrt[3]{\frac{\beta}{\gamma}} + \sqrt[3]{\frac{\gamma}{\alpha}} + \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{\gamma}{\beta}} + \sqrt[3]{\frac{\alpha}{\gamma}} = t = \sqrt[3]{\frac{AB}{C} - 9}.$$

□

Remark 4.5.2. *This result was originally due to Shevelev [77], where he provided a different proof.*

Corollary 4.5.3. *(Wituła [95]). If $f(x)$ is an $RCP2$, then the cubic Shevelev sum is*

$$\sqrt[3]{\frac{\alpha}{\beta}} + \sqrt[3]{\frac{\beta}{\gamma}} + \sqrt[3]{\frac{\gamma}{\alpha}} + \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{\gamma}{\beta}} + \sqrt[3]{\frac{\alpha}{\gamma}} = \sqrt[3]{\frac{1}{C}\left(A + 3\sqrt[3]{C}\right)\left(B + 3\sqrt[3]{C^2}\right)} + \sqrt[3]{\frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C} + B\right)}. \tag{4.55}$$

Proof. Let $f(x) = x^3 + Ax^2 + Bx + C = 0$ be a cubic equation satisfying Theorem 4.4.3. Then $t =$

$\sqrt[3]{\frac{1}{C}\left(A+3\sqrt[3]{C}\right)\left(B+3\sqrt[3]{C^2}\right)}+\sqrt[3]{\frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C}+B\right)}$. From Theorem 4.5.1, the cubic Shevelev sum is

$$\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}}=t$$
$$=\sqrt[3]{\frac{1}{C}\left(A+3\sqrt[3]{C}\right)\left(B+3\sqrt[3]{C^2}\right)}+\sqrt[3]{\frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C}+B\right)}.$$

□

Remark 4.5.3. *This result was due to Wituła [95] in which he provided a different proof.*

Corollary 4.5.4. *Let $f(x)=x^3-3rsx+rs(r+s)=0$ as described in Theorem 4.3.9. Then the cubic Shevelev sum is*

$$\sqrt[3]{\frac{\cos\left(\frac{\theta}{3}\right)}{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}}+\sqrt[3]{\frac{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}{\cos\left(\frac{\theta}{3}\right)}}+\sqrt[3]{\frac{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}}+\sqrt[3]{\frac{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}}+\sqrt[3]{\frac{\cos\left(\frac{\theta}{3}\right)}{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}}+\sqrt[3]{\frac{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}{\cos\left(\frac{\theta}{3}\right)}}$$
$$=\sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)+3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}+\sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)-3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}. \tag{4.56}$$

Proof. From Theorem 4.3.9, $t=\sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)+3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}+\sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)-3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}$. From Theorem 4.5.1, the cubic Shevelev sum is

$$\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}}=t$$
$$=\sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)+3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}+\sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)-3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}.$$

□

4.6 Computation Procedure

4.6.1 Type 1 & 2

In this section, two computation procedure are given. The first computation procedure is based on Wang's approach to finding the root of the associated Ramanujan equation $t^3+pt+q=0$. The second computation procedure is our approach. Again, denote $f(x) = x^3 - ax^2 + bx - 1$.

Find the roots of a cubic equation $f(x) = 0$: α, β, γ
$A_1 \leftarrow \alpha/\beta$
$A_2 \leftarrow \beta/\gamma$
$A_3 \leftarrow \gamma/\alpha$
$B \leftarrow A_1 + A_2 + A_3$
$C \leftarrow 1/A_1 + 1/A_2 + 1/A_3$
$D \leftarrow (B - C)^2$
$t \leftarrow \sqrt{D}$
return t

Given a cubic equation, Wang's computation procedure requires finding the roots of $f(x) = 0$ first. In the next three steps, we compute the ratios $\frac{\alpha}{\beta}, \frac{\beta}{\gamma}, \frac{\gamma}{\alpha}$, and their reciprocals. We sum these and compute the square of the difference. Taking the square root of this result will give the real root t.

$A \leftarrow ab + 6(a + b) + 9$
$B \leftarrow (ab)^2 - 4(a^3 + b^3) + 18ab - 27$
$C \leftarrow \sqrt{B}$
$D \leftarrow (A + C)/2$
$E \leftarrow (A - C)/2$
$t \leftarrow \sqrt[3]{D} + \sqrt[3]{E}$
return t

In our approach, we need only the coefficients a and b to compute the only real root t. Note that in the first step, $ab+6(a+b)+9$ is the negative of the constant term q of the associated Ramanujan equation. To obtain Ramanujan-type identities, we also need to find the roots of $f(x) = 0$ by using Vieta's formula. We will demonstrate the application of our computation procedure in the next section.

4.6.2 General Case

In this section, two computation procedures are given. The first is based on Theorems 4.2.1 and 4.3.7. This is to construct a Ramanujan-type identity given a general cubic equation $f(x) = x^3 + Ax^2 + Bx + C = 0$, with $A, B, C \in \mathbb{R}, C \neq 0$.

Computation procedure 1:

1: $M \leftarrow \dfrac{A^2 - 3B}{9}$.

2: $N \leftarrow \dfrac{2A^3 - 9AB + 27C}{3(A^2 - 3B)}$

3: Find the roots z_1, z_2 of $z^2 - Nz + M = 0$

4: $\theta \leftarrow \mathrm{Arg}(z_1)$

5: $\alpha \leftarrow -2\sqrt{M}\cos\dfrac{\theta}{3} - \dfrac{A}{3}$

6: $\beta \leftarrow -2\sqrt{M}\cos\left(\dfrac{\theta}{3} + \dfrac{2\pi}{3}\right) - \dfrac{A}{3}$

7: $\gamma \leftarrow -2\sqrt{M}\cos\left(\dfrac{\theta}{3} + \dfrac{4\pi}{3}\right) - \dfrac{A}{3}$

8: $q \leftarrow \sqrt{\dfrac{1}{C^2}\left((AB)^2 - 4(A^3C + B^3) + 18ABC - 27C^2\right)}$

9: $p \leftarrow \dfrac{AB}{C} + 6\left(\dfrac{A}{\sqrt[3]{C}} + \dfrac{B}{\sqrt[3]{C^2}}\right) + 9$

10: $t \leftarrow \sqrt[3]{\dfrac{p+q}{2}} + \sqrt[3]{\dfrac{p-q}{2}}$

11: $G_1 \leftarrow -\left(A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}\right)$

12: $G_2 \leftarrow -\dfrac{1}{C}\left(B + 6\sqrt[3]{C^2} + 3t\sqrt[3]{C^2}\right)$

13: Return the Ramanujan-type identities $\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{G_1}$ and $\dfrac{1}{\sqrt[3]{\alpha}} + \dfrac{1}{\sqrt[3]{\beta}} + \dfrac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{G_2}$.

Given a cubic equation, we first find the roots with the first 7 steps. Steps $11 - 12$ give the right hand sides of (4.23) and (4.24), respectively. Finally, step 13 returns the Ramanujan-type identities.

The second computation procedure gives the approach based on Theorem 4.3.9 for constructing a cosine Ramanujan-type identity. Given a complex number, we can the cubic equation and the Ramanujan-type identities.

Computation procedure 2:

1: Determine the cubic equation $x^3 - 3rsx + rs(r+s) = 0$.

2: $\alpha \leftarrow \cos\left(\dfrac{\theta}{3}\right)$

$\beta \leftarrow \cos\left(\dfrac{\theta}{3} + \dfrac{2\pi}{3}\right)$

$\gamma \leftarrow \cos\left(\dfrac{\theta}{3} + \dfrac{4\pi}{3}\right)$

$p \leftarrow \dfrac{3\sqrt{3}\eta}{\xi}$

$q \leftarrow 9\left(1 - \sqrt[3]{\dfrac{2|r|}{\xi^2}}\right)$

$t \leftarrow \sqrt[3]{\dfrac{q+p}{2}} + \sqrt[3]{\dfrac{q-p}{2}}$

$G_1 \leftarrow \sqrt[3]{\dfrac{2\xi}{|r|}}\left(3 + \dfrac{3t}{2}\right)$

$G_2 \leftarrow \left(\dfrac{3}{\xi}\right)\left(-|r| + 2\sqrt[3]{4\xi^2|r|} + t\sqrt[3]{4\xi^2|r|}\right)$

Return the cosine Ramanujan-type identities $\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{G_1}$ and $\dfrac{1}{\sqrt[3]{\alpha}} + \dfrac{1}{\sqrt[3]{\beta}} + \dfrac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{G_2}$.

4.7 Examples

4.7.1 Type 1

Example 4.7.1. (Wang [90]). *Let* $f(x) = x^3 - \sqrt[3]{9}x - 1 = 0$. *Denote* $a = 0$ *and* $b = -\sqrt[3]{9}$. *The associated Ramanujan equation is* $t^3 + pt + q = 0$, *where* $p = 3\sqrt[3]{9} - 9$ *and* $q = 6\sqrt[3]{9} - 9$. *Following the steps in Algorithm 4.6.1, then*

$$ab + 6(a+b) + 9 = A = -6\sqrt[3]{9} + 9.$$

Next,

$$\Delta^2 = B = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = -4(-9) - 27 = 9$$
$$\implies \Delta = C = 3,$$

and

$$D = \frac{1}{2}(A + C) = \frac{1}{2}\left(-6\sqrt[3]{9} + 9 + 3\right) = -3\sqrt[3]{9} + 6,$$
$$E = \frac{1}{2}(A - C) = \frac{1}{2}\left(-6\sqrt[3]{9} + 9 - 3\right) = -3\sqrt[3]{9} + 3.$$

Hence,

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{6 - 3\sqrt[3]{9}} + \sqrt[3]{3 - 3\sqrt[3]{9}}.$$

The roots of the cubic equation $x^3 - \sqrt[3]{9}x - 1 = 0$*, proved in Wang [90], are* $\alpha = -\frac{2}{\sqrt[6]{3}} \sin \frac{\pi}{9}$, $\beta = -\frac{2}{\sqrt[6]{3}} \sin \frac{2\pi}{9}$*, and* $\gamma = -\frac{2}{\sqrt[6]{3}} \sin \frac{14\pi}{9}$*. Then by Theorem 4.1.1,*

$$\sqrt[3]{\sin \frac{\pi}{9}} + \sqrt[3]{\sin \frac{2\pi}{9}} + \sqrt[3]{\sin \frac{14\pi}{9}} = -\frac{\sqrt[18]{3}}{2} \sqrt[3]{6 + 3\left(\sqrt[3]{6 - 3\sqrt[3]{9}} + \sqrt[3]{3 - 3\sqrt[3]{9}}\right)},$$

$$\frac{1}{\sqrt[3]{\sin \frac{\pi}{9}}} + \frac{1}{\sqrt[3]{\sin \frac{2\pi}{9}}} + \frac{1}{\sqrt[3]{\sin \frac{14\pi}{9}}} = -\frac{2}{\sqrt[18]{3}} \sqrt[3]{-\sqrt[3]{9} + 6 + 3\left(\sqrt[3]{6 - 3\sqrt[3]{9}} + \sqrt[3]{3 - 3\sqrt[3]{9}}\right)}.$$

Example 4.7.2. (Wang [90]). *Let* $f(x) = x^3 + 3\sqrt[3]{3}x^2 - \sqrt[3]{9}x - 1 = 0$*. Denote* $a = -3\sqrt[3]{3}$ *and* $b = -\sqrt[3]{9}$*. The associated Ramanujan equation is* $t^3 + pt + q = 0$*, where* $p = 3\left(3\sqrt[3]{3} + \sqrt[3]{9} - 3\right)$ *and* $q = -\left(18 - 6\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)\right)$*. Following the steps in Algorithm 4.6.1, then*

$$\begin{aligned} ab + 6(a+b) + 9 = A &= -3\sqrt[3]{3}\left(-\sqrt[3]{9}\right) + 6\left(-3\sqrt[3]{3} - \sqrt[3]{9}\right) + 9 \\ &= 18 - 6\left(3\sqrt[3]{3} + \sqrt[3]{9}\right). \end{aligned}$$

Next,

$$\begin{aligned} &\Delta^2 = B = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = 81 + 4(81 + 9) + 162 - 27 = 576 \\ \implies &\Delta = C = 24, \end{aligned}$$

and

$$\begin{aligned} D &= \frac{1}{2}(A + C) = \frac{1}{2}\left(18 - 6\left(3\sqrt[3]{3} + \sqrt[3]{9}\right) + 24\right) = 21 - 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right), \\ E &= \frac{1}{2}(A - C) = \frac{1}{2}\left(18 - 6\left(3\sqrt[3]{3} + \sqrt[3]{9}\right) - 24\right) = -3 - 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right). \end{aligned}$$

Hence,

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{21 - 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)} + \sqrt[3]{-3 - 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)}.$$

The roots of $x^3 + 3\sqrt[3]{3}x^2 - \sqrt[3]{9}x - 1 = 0$*, proved in Wang [90], are* $\alpha = -\frac{1}{\sqrt[6]{3}} \tan\frac{\pi}{9}$, $\beta = -\frac{1}{\sqrt[6]{3}} \tan\frac{4\pi}{9}$, *and* $\gamma = -\frac{1}{\sqrt[6]{3}} \tan\frac{7\pi}{9}$*. Then, by Theorem 4.1.1,*

$$\begin{aligned} &\sqrt[3]{\tan\frac{\pi}{9}} + \sqrt[3]{\tan\frac{4\pi}{9}} + \sqrt[3]{\tan\frac{7\pi}{9}} \\ &\qquad = \left(-\sqrt[18]{3}\right)\left(\sqrt[3]{-3\sqrt[3]{3} + 6 + 3\left(\sqrt[3]{21 - 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)} - \sqrt[3]{3 + 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)}\right)}\right), \\ &\frac{1}{\sqrt[3]{\tan\frac{\pi}{9}}} + \frac{1}{\sqrt[3]{\tan\frac{4\pi}{9}}} + \frac{1}{\sqrt[3]{\tan\frac{7\pi}{9}}} \\ &\qquad = \left(-\frac{1}{\sqrt[18]{3}}\right)\left(\sqrt[3]{-\sqrt[3]{9} + 6 + 3\left(\sqrt[3]{21 - 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)} - \sqrt[3]{3 + 3\left(3\sqrt[3]{3} + \sqrt[3]{9}\right)}\right)}\right). \end{aligned}$$

Example 4.7.3. (Wang [90]). *Let* $f(x) = x^3 + \sqrt[3]{7}x^2 - 1 = 0$*. Denote* $a = -\sqrt[3]{7}$ *and* $b = 0$*. The associated Ramanujan equation is* $t^3 + pt + q = 0$*, where* $p = 3\left(\sqrt[3]{7} - 3\right)$ *and* $q = 6\sqrt[3]{7} - 9$*. Following the steps in Algorithm 4.6.1, then*

$$ab + 6(a + b) + 9 = A = -6\sqrt[3]{7} + 9,$$

Next,

$$\Delta^2 = B = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = -4(-7) - 27 = 1 \implies \Delta = C = 1.$$

and

$$\begin{aligned} D &= \frac{1}{2}(A + C) = \frac{1}{2}\left(-6\sqrt[3]{7} + 9 + 1\right) = -3\sqrt[3]{7} + 5, \\ E &= \frac{1}{2}(A - C) = \frac{1}{2}\left(-6\sqrt[3]{7} + 9 - 1\right) = -3\sqrt[3]{7} + 4. \end{aligned}$$

Hence,

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{5 - 3\sqrt[3]{7}} + \sqrt[3]{4 - 3\sqrt[3]{7}}.$$

The roots of $x^3 + \sqrt[3]{7}x^2 - 1 = 0$*, proved in Wang [90], are* $\alpha = -\frac{2}{\sqrt[6]{7}} \sin \frac{2\pi}{7}$, $\beta = -\frac{2}{\sqrt[6]{7}} \sin \frac{4\pi}{7}$*, and* $\gamma = -\frac{2}{\sqrt[6]{7}} \sin \frac{8\pi}{7}$*. Then, by Theorem 4.1.1,*

$$\begin{aligned}
&\sqrt[3]{\sin \frac{2\pi}{7}} + \sqrt[3]{\sin \frac{4\pi}{7}} + \sqrt[3]{\sin \frac{8\pi}{7}} \\
&\qquad = \left(-\sqrt[18]{\frac{7}{64}} \right) \left(\sqrt[3]{-\sqrt[3]{7} + 6 + 3\left(\sqrt[3]{5 - 3\sqrt[3]{7}} + \sqrt[3]{4 - 3\sqrt[3]{7}} \right)} \right), \\
&\frac{1}{\sqrt[3]{\sin \frac{2\pi}{7}}} + \frac{1}{\sqrt[3]{\sin \frac{4\pi}{7}}} + \frac{1}{\sqrt[3]{\sin \frac{8\pi}{7}}} \\
&\qquad = \left(-\sqrt[18]{\frac{64}{7}} \right) \left(\sqrt[3]{6 + 3\left(\sqrt[3]{5 - 3\sqrt[3]{7}} + \sqrt[3]{4 - 3\sqrt[3]{7}} \right)} \right).
\end{aligned}$$

Example 4.7.4. (Wang [90]). *Let* $f(x) = x^3 - \sqrt[3]{7}x^2 - \sqrt[3]{49}x - 1 = 0$*. Denote* $a = \sqrt[3]{7}$ *and* $b = -\sqrt[3]{49}$*. The associated Ramanujan equation is* $t^3 + pt + q = 0$*, where* $p = -3\left(\sqrt[3]{7} - \sqrt[3]{49} + 3 \right)$ *and* $q = 6\sqrt[3]{7} - 6\sqrt[3]{49} + 2$*. Following the steps in Algorithm 4.6.1, then*

$$ab + 6(a+b) + 9 = A = \sqrt[3]{7}\left(-\sqrt[3]{49} \right) + 6\left(\sqrt[3]{7} - \sqrt[3]{49} \right) + 9 = 2 + 6\left(\sqrt[3]{7} - \sqrt[3]{49} \right).$$

Next,

$$\begin{aligned}
&\Delta^2 = B = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = 49 - 4(7 - 49) - 126 - 27 = 64 \\
\Longrightarrow\ &\Delta = C = 8,
\end{aligned}$$

and

$$\begin{aligned}
D &= \frac{1}{2}(A + C) = \frac{1}{2}\left(2 + 6\left(\sqrt[3]{7} - \sqrt[3]{49} \right) + 8 \right) = 5 + 3\left(\sqrt[3]{7} - \sqrt[3]{49} \right), \\
E &= \frac{1}{2}(A - C) = \frac{1}{2}\left(2 + 6\left(\sqrt[3]{7} - \sqrt[3]{49} \right) - 8 \right) = -3 + 3\left(\sqrt[3]{7} - \sqrt[3]{49} \right).
\end{aligned}$$

Hence,

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{5 + 3\left(\sqrt[3]{7} - \sqrt[3]{49}\right)} + \sqrt[3]{3\left(\sqrt[3]{7} - \sqrt[3]{49}\right) - 3}.$$

The roots of $x^3 - \sqrt[3]{7}x^2 - \sqrt[3]{49}x - 1 = 0$*, proved in Wang [90], are* $\alpha = -\frac{1}{\sqrt[6]{7}}\tan\frac{2\pi}{7}$*,* $\beta = -\frac{1}{\sqrt[6]{7}}\tan\frac{4\pi}{7}$*, and* $\gamma = -\frac{1}{\sqrt[6]{7}}\tan\frac{8\pi}{7}$*. Then, by Theorem 4.1.1,*

$$
\begin{aligned}
&\sqrt[3]{\tan\frac{2\pi}{7}} + \sqrt[3]{\tan\frac{4\pi}{7}} + \sqrt[3]{\tan\frac{8\pi}{7}} \\
&\qquad = \left(\sqrt[18]{7}\right)\sqrt[3]{-\sqrt[3]{7} + 6 + 3\left(\sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} + 5} + \sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} - 3}\right)}, \\
&\frac{1}{\sqrt[3]{\tan\frac{2\pi}{7}}} + \frac{1}{\sqrt[3]{\tan\frac{4\pi}{7}}} + \frac{1}{\sqrt[3]{\tan\frac{8\pi}{7}}} \\
&\qquad = \left(-\frac{1}{\sqrt[18]{7}}\right)\sqrt[3]{-\sqrt[3]{49} + 6 + 3\left(\sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} + 5} + \sqrt[3]{3\sqrt[3]{7} - 3\sqrt[3]{49} - 3}\right)}.
\end{aligned}
$$

The following cubic equations are selected from Wang's preprint Wang [89]. In his preprint, no Ramanujan-type identities and proof of roots were given. The proof of the roots are given in the Appendix. The authors complete the Ramanujan-type identities in the following examples.

Example 4.7.5. (Wang [89]). *Let* $f(x) = x^3 + 4x^2 + 3x - 1 = 0$*. Denote* $a = -4$ *and* $b = 3$*. The associated Ramanujan equation is* $t^3 + pt + q = 0$*, where* $p = -6$ *and* $q = 9$*. Following the steps in Algorithm 4.6.1, then*

$$ab + 6(a + b) + 9 = A = -4(3) + 6(-4 + 3) + 9 = -9.$$

Next,

$$\Delta^2 = B = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = 49 \implies \Delta = C = 7.$$

and

$$
\begin{aligned}
D &= \frac{1}{2}(A + C) = \frac{1}{2}(-9 + 7) = -1, \\
E &= \frac{1}{2}(A - C) = \frac{1}{2}(-9 - 7) = -8.
\end{aligned}
$$

Then

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{-1} + \sqrt[3]{-8} = -3.$$

The roots of $x^3 + 4x^2 + 3x - 1 = 0$, *proved in the Appendix, are* $\alpha = -\frac{8}{\sqrt{7}}\sin^3\left(\frac{2\pi}{7}\right)$, $\beta = -\frac{8}{\sqrt{7}}\sin^3\left(\frac{4\pi}{7}\right)$, *and* $\gamma = -\frac{8}{\sqrt{7}}\sin^3\left(\frac{8\pi}{7}\right)$. *Then, by Theorem 4.1.1,*

$$\begin{aligned}\sqrt[3]{-\frac{8}{\sqrt{7}}\sin^3\left(\frac{2\pi}{7}\right)} + \sqrt[3]{-\frac{8}{\sqrt{7}}\sin^3\left(\frac{4\pi}{7}\right)} + \sqrt[3]{-\frac{8}{\sqrt{7}}\sin^3\left(\frac{8\pi}{7}\right)} &= \sqrt[3]{-4+6+3(-3)} \\ \Longrightarrow \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right) &= \frac{\sqrt{7}}{2}. \end{aligned} \tag{4.57}$$

$$\begin{aligned}\frac{1}{\sqrt[3]{-\frac{8}{\sqrt{7}}\sin^3\left(\frac{2\pi}{7}\right)}} + \frac{1}{\sqrt[3]{-\frac{8}{\sqrt{7}}\sin^3\left(\frac{4\pi}{7}\right)}} + \frac{1}{\sqrt[3]{-\frac{8}{\sqrt{7}}\sin^3\left(\frac{8\pi}{7}\right)}} &= \sqrt[3]{3+6+3(-3)} \\ \Longrightarrow \csc\left(\frac{2\pi}{7}\right) + \csc\left(\frac{4\pi}{7}\right) + \csc\left(\frac{8\pi}{7}\right) &= 0. \end{aligned} \tag{4.58}$$

Example 4.7.6. (Wang [89]). *Let* $f(x) = x^3 + 4x^2 + 3x - 1 = 0$. *Denote* $a = -4$ *and* $b = 3$. *The associated Ramanujan equation is* $t^3 + pt + q = 0$, *where* $p = -6$ *and* $q = 9$. *From the previous example, we have* $t = -3$. *The roots of* $x^3 + 4x^2 + 3x - 1 = 0$, *proved in the Appendix, are* $\alpha = \frac{\cos\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}$, $\beta = \frac{\cos\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}$, *and* $\gamma = \frac{\cos\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}$. *Then, by Theorem 4.1.1,*

$$\sqrt[3]{\frac{\cos\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}} = -\sqrt[3]{7}, \tag{4.59}$$

$$\sqrt[3]{\frac{\cos\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}} = 0. \tag{4.60}$$

Example 4.7.7. (Wang [89]). *Let* $f(x) = x^3 + 46x^2 + 3x - 1 = 0$. *Denote* $a = -46$ *and* $b = 3$. *The associated Ramanujan equation is* $t^3 + pt + q = 0$, *where* $p = 120$ *and* $q = 387$. *Following the* *steps in Algorithm 4.6.1, then*

$$ab + 6(a+b) + 9 = A = -46(3) + 6(-46+3) + 9 = -387.$$

Next,

$$\Delta^2 = B = (ab)^2 - 4(a^3+b^3) + 18ab - 27 = 405769 \implies \Delta = C = 637,$$

and

$$\begin{aligned} D &= \frac{1}{2}(A+C) = \frac{1}{2}(-387+637) = 125, \\ E &= \frac{1}{2}(A-C) = \frac{1}{2}(-387-637) = -512. \end{aligned}$$

Hence,

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{125} + \sqrt[3]{-512} = -3.$$

The roots of $x^3 + 46x^2 + 3x - 1 = 0$, *proved in the Appendix, are* $\alpha = -\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^2\left(\frac{4\pi}{7}\right)}\right)\right)^3$, $\beta = -\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^2\left(\frac{8\pi}{7}\right)}\right)\right)^3$, *and* $\gamma = -\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^2\left(\frac{2\pi}{7}\right)}\right)\right)^3$. *Then by Theorem 4.1.1,*

$$\begin{aligned} &\sqrt[3]{-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^2\left(\frac{4\pi}{7}\right)}\right)\right)^3} + \sqrt[3]{-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^2\left(\frac{8\pi}{7}\right)}\right)\right)^3} + \sqrt[3]{-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^2\left(\frac{2\pi}{7}\right)}\right)\right)^3} \\ &= \sqrt[3]{-46+6+3(-3)} \\ &\implies \frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^2\left(\frac{4\pi}{7}\right)} + \frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^2\left(\frac{8\pi}{7}\right)} + \frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^2\left(\frac{2\pi}{7}\right)} = 2\sqrt{7}, \qquad (4.61) \\ &\frac{1}{\sqrt[3]{-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^2\left(\frac{4\pi}{7}\right)}\right)\right)^3}} + \frac{1}{\sqrt[3]{-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^2\left(\frac{8\pi}{7}\right)}\right)\right)^3}} + \frac{1}{\sqrt[3]{-\left(\frac{\sqrt[6]{7}}{2}\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^2\left(\frac{2\pi}{7}\right)}\right)\right)^3}} \\ &= \sqrt[3]{3+6+3(-3)} \end{aligned}$$

$$\Longrightarrow \frac{\sin^2\left(\frac{4\pi}{7}\right)}{\sin\left(\frac{2\pi}{7}\right)} + \frac{\sin^2\left(\frac{8\pi}{7}\right)}{\sin\left(\frac{4\pi}{7}\right)} + \frac{\sin^2\left(\frac{2\pi}{7}\right)}{\sin\left(\frac{8\pi}{7}\right)} = 0. \tag{4.62}$$

Example 4.7.8. (Wang [89]). *Let $f(x) = x^3 - 3x^2 - 46x - 1 = 0$. Denote $a = 3$ and $b = -46$. The associated Ramanujan equation is $t^3 + pt + q = 0$, where $p = 120$ and $q = 387$. From the last example,*

$$t = -3.$$

The roots of $x^3 - 3x^2 - 46x - 1 = 0$, proved in the Appendix, are $\alpha = 2^3\left(\frac{\cos^4\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}\right)$, $\beta = 2^3\left(\frac{\cos^4\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}\right)$, and $\gamma = 2^3\left(\frac{\cos^4\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}\right)$. Then by Theorem 4.1.1,

$$\sqrt[3]{2^3\left(\frac{\cos^4\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}\right)} + \sqrt[3]{2^3\left(\frac{\cos^4\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}\right)} + \sqrt[3]{2^3\left(\frac{\cos^4\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}\right)} = \sqrt[3]{3 + 6 + 3(-3)}$$

$$\Longrightarrow \sqrt[3]{\frac{\cos^4\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^4\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^4\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}} = 0, \tag{4.63}$$

$$\frac{1}{\sqrt[3]{2^3\left(\frac{\cos^4\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{4\pi}{7}\right)}\right)}} + \frac{1}{\sqrt[3]{2^3\left(\frac{\cos^4\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{8\pi}{7}\right)}\right)}} + \frac{1}{\sqrt[3]{2^3\left(\frac{\cos^4\left(\frac{8\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}\right)}} = \sqrt[3]{-46 + 6 + 3(-3)}$$

$$\Longrightarrow \sqrt[3]{\frac{\cos\left(\frac{4\pi}{7}\right)}{\cos^4\left(\frac{2\pi}{7}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{8\pi}{7}\right)}{\cos^4\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{2\pi}{7}\right)}{\cos^4\left(\frac{8\pi}{7}\right)}} = -2\sqrt[3]{49}. \tag{4.64}$$

Example 4.7.9. (Wang [89]). *Let $f(x) = x^3 + 186x^2 + 3x - 1 = 0$. Denote $a = -186$ and $b = 3$. The associated Ramanujan equation is $t^3 + pt + q = 0$, where $p = 540$ and $q = 1647$. Following the steps in Algorithm 4.6.1, then*

$$ab + 6(a + b) + 9 = A = -186(3) + 6(-186 + 3) + 9 = -1647.$$

Next,

$$\Delta^2 = B = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = 26040609 \Longrightarrow \Delta = C = 5103.$$

and

$$D = \frac{1}{2}(A + C) = \frac{1}{2}(-1647 + 5103) = 1728,$$
$$E = \frac{1}{2}(A - C) = \frac{1}{2}(-1647 - 5103) = 3375.$$

Hence,

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{1728} + \sqrt[3]{3375} = 12 - 15 = -3.$$

The roots of $x^3 + 186x^2 + 3x - 1 = 0$*, proved in the Appendix, are* $\alpha = \left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^3\left(\frac{4\pi}{7}\right)}\right)\right)^3$, $\beta = \left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^3\left(\frac{8\pi}{7}\right)}\right)\right)^3$*, and* $\gamma = \left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^3\left(\frac{2\pi}{7}\right)}\right)\right)^3$*. Then by Theorem 4.1.1,*

$$\sqrt[3]{\left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^3\left(\frac{4\pi}{7}\right)}\right)\right)^3} + \sqrt[3]{\left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^3\left(\frac{8\pi}{7}\right)}\right)\right)^3} + \sqrt[3]{\left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^3\left(\frac{2\pi}{7}\right)}\right)\right)^3}$$
$$= \sqrt[3]{-186 + 6 + 3(-3)}$$
$$\Longrightarrow \frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^3\left(\frac{4\pi}{7}\right)} + \frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^3\left(\frac{8\pi}{7}\right)} + \frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^3\left(\frac{2\pi}{7}\right)} = -12, \tag{4.65}$$

$$\frac{1}{\sqrt[3]{\left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\sin^3\left(\frac{4\pi}{7}\right)}\right)\right)^3}} + \frac{1}{\sqrt[3]{\left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{\sin^3\left(\frac{8\pi}{7}\right)}\right)\right)^3}} + \frac{1}{\sqrt[3]{\left(\frac{\sqrt[3]{7}}{4}\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin^3\left(\frac{2\pi}{7}\right)}\right)\right)^3}}$$
$$= \sqrt[3]{3 + 6 + 3(-3)}$$
$$\Longrightarrow \frac{\sin^3\left(\frac{4\pi}{7}\right)}{\sin\left(\frac{2\pi}{7}\right)} + \frac{\sin^3\left(\frac{8\pi}{7}\right)}{\sin\left(\frac{4\pi}{7}\right)} + \frac{\sin^3\left(\frac{2\pi}{7}\right)}{\sin\left(\frac{8\pi}{7}\right)} = 0. \tag{4.66}$$

Example 4.7.10. (Wang [89]). *Let* $f(x) = x^3 - 3x^2 - 186x - 1 = 0$*. Denote* $a = 3$ *and* $b = -186$*. The associated Ramanujan equation is* $t^3 + pt + q = 0$*, where* $p = 540$ *and* $q = 1647$*. From the last example*

$$t = -3.$$

The roots of $x^3 - 3x^2 - 186x - 1 = 0$*, proved in the Appendix, are* $\alpha = 2^3\left(\frac{\cos^5\left(\frac{2\pi}{7}\right)}{\cos^2\left(\frac{4\pi}{7}\right)}\right)$,

$\beta = 2^3 \left(\frac{\cos^5\left(\frac{4\pi}{7}\right)}{\cos^2\left(\frac{8\pi}{7}\right)} \right)$, *and* $\gamma = 2^3 \left(\frac{\cos^5\left(\frac{8\pi}{7}\right)}{\cos^2\left(\frac{2\pi}{7}\right)} \right)$. *Then by Theorem 4.1.1,*

$$\sqrt[3]{2^3 \left(\frac{\cos^5\left(\frac{2\pi}{7}\right)}{\cos^2\left(\frac{4\pi}{7}\right)} \right)} + \sqrt[3]{2^3 \left(\frac{\cos^5\left(\frac{4\pi}{7}\right)}{\cos^2\left(\frac{8\pi}{7}\right)} \right)} + \sqrt[3]{2^3 \left(\frac{\cos^5\left(\frac{8\pi}{7}\right)}{\cos^2\left(\frac{2\pi}{7}\right)} \right)} = \sqrt{3+6+3(-3)}$$

$$\implies \sqrt[3]{\frac{\cos^5\left(\frac{2\pi}{7}\right)}{\cos^2\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^5\left(\frac{4\pi}{7}\right)}{\cos^2\left(\frac{8\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^5\left(\frac{8\pi}{7}\right)}{\cos^2\left(\frac{2\pi}{7}\right)}} = 0, \tag{4.67}$$

$$\frac{1}{\sqrt[3]{2^3 \left(\frac{\cos^5\left(\frac{2\pi}{7}\right)}{\cos^2\left(\frac{4\pi}{7}\right)} \right)}} + \frac{1}{\sqrt[3]{2^3 \left(\frac{\cos^5\left(\frac{4\pi}{7}\right)}{\cos^2\left(\frac{8\pi}{7}\right)} \right)}} + \frac{1}{\sqrt[3]{2^3 \left(\frac{\cos^5\left(\frac{8\pi}{7}\right)}{\cos^2\left(\frac{2\pi}{7}\right)} \right)}} = \sqrt[3]{-186+6+3(-3)}$$

$$\implies \sqrt[3]{\frac{\cos^2\left(\frac{4\pi}{7}\right)}{\cos^5\left(\frac{2\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^2\left(\frac{8\pi}{7}\right)}{\cos^5\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^2\left(\frac{2\pi}{7}\right)}{\cos^5\left(\frac{8\pi}{7}\right)}} = -6\sqrt[3]{7}. \tag{4.68}$$

Example 4.7.11. (Wang [89]). *Let* $f(x) = x^3 - 3x^2 - 1588864x - 1 = 0$. *Denote* $a = 3$ *and* $b = -1588864$. *The associated Ramanujan equation is* $t^3 + pt + q = 0$, *where* $p = 4766574$ *and* $q = 14299359$. *Following the steps in Algorithm 4.6.1, then*

$$ab + 6(a+b) + 9 = A = 3(-1588864) + 6(a - 1588864) + 9 = -14299359.$$

Next,

$$\Delta^2 = B = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 = 16044300277913161849$$
$$\implies \Delta = C = 4005533707,$$

and

$$D = \frac{1}{2}(A+C) = \frac{1}{2}(-14299359 + 4005533707) = 1995617174,$$
$$E = \frac{1}{2}(A-C) = \frac{1}{2}(-14299359 - 4005533707) = -2009916533.$$

Then

$$t = \sqrt[3]{D} + \sqrt[3]{E} = \sqrt[3]{1995617174} + \sqrt[3]{-2009916533} = -3.$$

The roots of $x^3 - 3x^2 - 1588864x - 1 = 0$*, proved in the Appendix, are* $\alpha = 2^9\left(\frac{\cos^{14}\left(\frac{2\pi}{7}\right)}{\cos^5\left(\frac{4\pi}{7}\right)}\right)$, $\beta = 2^9\left(\frac{\cos^{14}\left(\frac{4\pi}{7}\right)}{\cos^5\left(\frac{8\pi}{7}\right)}\right)$*, and* $\gamma = 2^9\left(\frac{\cos^{14}\left(\frac{8\pi}{7}\right)}{\cos^5\left(\frac{2\pi}{7}\right)}\right)$*. Then by Theorem 4.1.1,*

$$\sqrt[3]{2^9\left(\frac{\cos^{14}\left(\frac{2\pi}{7}\right)}{\cos^5\left(\frac{4\pi}{7}\right)}\right)} + \sqrt[3]{2^9\left(\frac{\cos^{14}\left(\frac{4\pi}{7}\right)}{\cos^5\left(\frac{8\pi}{7}\right)}\right)} + \sqrt[3]{2^9\left(\frac{\cos^{14}\left(\frac{8\pi}{7}\right)}{\cos^5\left(\frac{2\pi}{7}\right)}\right)} = \sqrt[3]{3+6+3(-3)}$$

$$\implies \sqrt[3]{\frac{\cos^{14}\left(\frac{2\pi}{7}\right)}{\cos^5\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^{14}\left(\frac{4\pi}{7}\right)}{\cos^5\left(\frac{8\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^{14}\left(\frac{8\pi}{7}\right)}{\cos^5\left(\frac{2\pi}{7}\right)}} = 0, \tag{4.69}$$

$$\frac{1}{\sqrt[3]{2^9\left(\frac{\cos^{14}\left(\frac{2\pi}{7}\right)}{\cos^5\left(\frac{4\pi}{7}\right)}\right)}} + \frac{1}{\sqrt[3]{2^9\left(\frac{\cos^{14}\left(\frac{4\pi}{7}\right)}{\cos^5\left(\frac{8\pi}{7}\right)}\right)}} + \frac{1}{\sqrt[3]{2^9\left(\frac{\cos^{14}\left(\frac{8\pi}{7}\right)}{\cos^5\left(\frac{2\pi}{7}\right)}\right)}} = \sqrt[3]{-1588864+6+3(-3)}$$

$$\implies \sqrt[3]{\frac{\cos^{5}\left(\frac{4\pi}{7}\right)}{\cos^{14}\left(\frac{2\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^{5}\left(\frac{8\pi}{7}\right)}{\cos^{14}\left(\frac{4\pi}{7}\right)}} + \sqrt[3]{\frac{\cos^{5}\left(\frac{2\pi}{7}\right)}{\cos^{14}\left(\frac{8\pi}{7}\right)}} = -488\sqrt[3]{7}. \tag{4.70}$$

4.7.2 Type 2

In this section, we give examples on the cubic equation $x^3 - ax^2 + bx - 1 = 0$ with the condition $a + b + 3 = 0$. This condition has been heavily studied in literature, such as Shevelev [77], Dresden et al. [22], Berndt and Bhargava [7].

Example 4.7.12. *Let* $f(x) = x^3 + x^2 - 2x - 1 = 0$*. Denote* $a = -1$ *and* $b = -2$*. The associated Ramanujan equation is* $t^3 + 7 = 0$*. Hence,* $t = -\sqrt[3]{7}$*. The roots of* $x^3 + x^2 - 2x - 1 = 0$ *are* $\alpha = 2\cos\frac{2\pi}{7}$*,* $\beta = 2\cos\frac{4\pi}{7}$*, and* $\gamma = 2\cos\frac{8\pi}{7}$ *(Liao et al. [50]). By Theorem 4.1.1, we have the following Ramanujan-type identities:*

$$\sqrt[3]{2\cos\frac{2\pi}{7}} + \sqrt[3]{2\cos\frac{4\pi}{7}} + \sqrt[3]{2\cos\frac{8\pi}{7}} = \sqrt[3]{-1+6-3\sqrt[3]{7}}$$

$$\implies \sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{8\pi}{7}} = \sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}},$$

and

$$\frac{1}{\sqrt[3]{2\cos\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{2\cos\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{2\cos\frac{8\pi}{7}}}=\sqrt[3]{-2+6-3\sqrt[3]{7}}$$
$$\implies \frac{1}{\sqrt[3]{\cos\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{\cos\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{\cos\frac{8\pi}{7}}}=\sqrt[3]{8-6\sqrt[3]{7}}.$$

Example 4.7.13. *Let* $f(x)=x^3+\dfrac{3}{2}x^2-\dfrac{3}{2}x-1=0$. *Denote* $a=-\dfrac{3}{2}$ *and* $b=-\dfrac{3}{2}$, *i.e.,* $a+b+3=0$. *By Corollary 4.3.3,* $t=\sqrt[3]{\dfrac{9}{4}-9}=-\dfrac{3}{\sqrt[3]{4}}$.

Using rational roots theorem, the roots of $f(x)=0$ *are* $\alpha=-2$,$\beta=-\dfrac{1}{2}$, *and* $\gamma=1$. *Thus,*

$$\sqrt[3]{-2}+\sqrt[3]{-\frac{1}{2}}+\sqrt[3]{1}=\sqrt[3]{-\frac{3}{2}+6-\frac{9}{\sqrt[3]{4}}}$$
$$\implies \sqrt[3]{2}-\sqrt[3]{4}-1=\sqrt[3]{9-\frac{18}{\sqrt[3]{4}}}.$$

Example 4.7.14. *Let* $f(x)=x^3-\dfrac{3}{4}x+\dfrac{1}{8}=0$. *It is known that the roots are* $\alpha=\cos\dfrac{2\pi}{9}$, $\beta=\cos\dfrac{4\pi}{9}$, *and* $\gamma=\cos\dfrac{8\pi}{9}$ *(Liao et al. [50]). Apply the transformation* $x=-\dfrac{1}{2}y$, *then we obtain*

$$-\frac{1}{8}y^3+\frac{3}{4}\cdot\frac{1}{2}y+\frac{1}{8}=0 \implies g(y)=y^3-3y-1=0.$$

Denote $a=0$ *and* $b=-3$, *i.e.,* $a+b+3=0$. *Then by Corollary 4.3.3, we have* $t=\sqrt[3]{-9}$. *Using the transformation, the roots of* $g(y)=0$ *are* $y=-2\cos\dfrac{2\pi}{9}$, $-2\cos\dfrac{4\pi}{9}$, *and* $-2\cos\dfrac{8\pi}{9}$. *Thus,*

$$\sqrt[3]{-2\cos\frac{2\pi}{9}}+\sqrt[3]{-2\cos\frac{4\pi}{9}}+\sqrt[3]{-2\cos\frac{8\pi}{9}}=\sqrt[3]{6-3\sqrt[3]{9}}$$
$$\implies \sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=-\sqrt[3]{\frac{6-3\sqrt[3]{9}}{2}},$$

and

$$\frac{1}{\sqrt[3]{-2\cos\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{-2\cos\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{-2\cos\frac{8\pi}{9}}}=\sqrt[3]{3-3\sqrt[3]{9}}$$

$$\Longrightarrow \frac{1}{\sqrt[3]{\cos\frac{2\pi}{9}}} + \frac{1}{\sqrt[3]{\cos\frac{4\pi}{9}}} + \frac{1}{\sqrt[3]{\cos\frac{8\pi}{9}}} = -\sqrt[3]{6 - 6\sqrt[3]{9}}.$$

4.7.3 RCP

The following cubic equation is from Wituła [95]:

Example 4.7.15. *Let* $f(x) = x^3 + 7x^2 - 98x - 343 = 0$. *Denote* $A = -7$, $B = -98$, *and* $C = 343$. *The roots of* $x^3 + 7x^2 - 98x - 343 = 0$ *are* $\alpha = 14\cos\frac{2\pi}{7}$, $\beta = 14\cos\frac{4\pi}{7}$, *and* $\gamma = 14\cos\frac{8\pi}{7}$ *(Wituła [95]). Note that* $f(x)$ *is an RCP. Using* (4.22),

$$\frac{D(f)}{C^2} = \left(\frac{7\cdot 98}{343} - 9\right)^2 = (2-9)^2.$$

Then

$$\frac{\sqrt{D(f)}}{C} = 7.$$

Next, by (4.44),

$$t = -\sqrt[3]{7}.$$

Thus, by (4.45),

$$\sqrt[3]{14\cos\frac{2\pi}{7}} + \sqrt[3]{14\cos\frac{4\pi}{7}} + \sqrt[3]{14\cos\frac{8\pi}{7}} = 343^{\frac{1}{9}}\sqrt[3]{-1 + 6 - 3\sqrt[3]{7}},$$

which simplifies to

$$\sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{8\pi}{7}} = \sqrt[3]{\frac{5 - 3\sqrt[3]{7}}{2}}.$$

By (4.46),

$$\frac{1}{\sqrt[3]{14\cos\frac{2\pi}{7}}} + \frac{1}{\sqrt[3]{14\cos\frac{4\pi}{7}}} + \frac{1}{\sqrt[3]{14\cos\frac{8\pi}{7}}} = \frac{1}{343^{\frac{1}{9}}}\sqrt[3]{-2 + 6 - 3\sqrt[3]{7}},$$

which simplifies to

$$\frac{1}{\sqrt[3]{\cos\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{\cos\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{\cos\frac{8\pi}{7}}}=\sqrt[3]{8-6\sqrt[3]{7}}.$$

Using (4.54)*, the cubic Shevelev sum is*

$$\sqrt[3]{\frac{\cos\frac{2\pi}{7}}{\cos\frac{4\pi}{7}}}+\sqrt[3]{\frac{\cos\frac{4\pi}{7}}{\cos\frac{2\pi}{7}}}+\sqrt[3]{\frac{\cos\frac{2\pi}{7}}{\cos\frac{8\pi}{7}}}+\sqrt[3]{\frac{\cos\frac{8\pi}{7}}{\cos\frac{2\pi}{7}}}+\sqrt[3]{\frac{\cos\frac{4\pi}{7}}{\cos\frac{8\pi}{7}}}+\sqrt[3]{\frac{\cos\frac{8\pi}{7}}{\cos\frac{4\pi}{7}}}=-\sqrt[3]{7}. \tag{4.71}$$

Example 4.7.16. *(Shevelev [77]). Let* $f(x)=x^3-\frac{3}{4}x+\frac{1}{8}=0$*. Denote* $A=0$*,* $B=-\frac{3}{4}$*, and* $C=\frac{1}{8}$*. It is known that the roots are* $\alpha=\cos\frac{2\pi}{9}$*,* $\beta=\cos\frac{4\pi}{9}$*, and* $\gamma=\cos\frac{8\pi}{9}$*. Note that* $f(x)$ *is an RCP. Using* (4.44)*,*

$$t=\sqrt[3]{\frac{AB}{C}-9}=-\sqrt[3]{9}.$$

The Ramanujan-type identities are obtained by using (4.45) *and* (4.46)*:*

$$\sqrt[3]{-2\cos\frac{2\pi}{9}}+\sqrt[3]{-2\cos\frac{4\pi}{9}}+\sqrt[3]{-2\cos\frac{8\pi}{9}}=\sqrt[3]{6-3\sqrt[3]{9}},$$

which simplifies to

$$\sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=-\sqrt[3]{\frac{6-3\sqrt[3]{9}}{2}}$$

On the other hand,

$$\frac{1}{\sqrt[3]{-2\cos\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{-2\cos\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{-2\cos\frac{8\pi}{9}}}=\sqrt[3]{3-3\sqrt[3]{9}},$$

which simplifies to

$$\frac{1}{\sqrt[3]{\cos\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{\cos\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{\cos\frac{8\pi}{9}}}=-\sqrt[3]{6-6\sqrt[3]{9}}.$$

By (4.54)*, the cubic Shevelev sum is*

$$\sqrt[3]{\frac{\cos\frac{2\pi}{9}}{\cos\frac{4\pi}{9}}}+\sqrt[3]{\frac{\cos\frac{4\pi}{9}}{\cos\frac{2\pi}{9}}}+\sqrt[3]{\frac{\cos\frac{2\pi}{9}}{\cos\frac{8\pi}{9}}}+\sqrt[3]{\frac{\cos\frac{8\pi}{9}}{\cos\frac{2\pi}{9}}}+\sqrt[3]{\frac{\cos\frac{4\pi}{9}}{\cos\frac{8\pi}{9}}}+\sqrt[3]{\frac{\cos\frac{8\pi}{9}}{\cos\frac{4\pi}{9}}}=-\sqrt[3]{9}.$$

Example 4.7.17. *Let* $f(x)=x^3+x^2-(3n^2+n)x+n^3=0,\ n\in\mathbb{N}$. *Denote* $A=1$, $B=-(3n^2+n)$, *and* $C=n^3$. *Note that this is an RCP. By Theorem 4.4.1, we have*

$$9C-AB=9n^3+3n^2+n=n(9n^2+3n+1).$$

Denote the roots of $f(x)$ *as* α, β, *and* γ, *then by* (4.23),

$$\begin{aligned}\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}&=\sqrt[3]{-\left(A+6\sqrt[3]{C}\right)+3\sqrt[3]{9C-AB}}\\&=\sqrt[3]{-(6n+1)+3\sqrt[3]{n(9n^2+3n+1)}}.\end{aligned}$$

Then with (4.24),

$$\begin{aligned}\frac{1}{\sqrt[3]{\alpha}}+\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}}&=\frac{1}{\sqrt[3]{C}}\sqrt[3]{-B-6\sqrt[3]{C^2}+3\sqrt[3]{9C^2-ABC}}\\&=\frac{1}{n}\sqrt[3]{3n^2+n-6n^2+3\sqrt[3]{n^4\left(9n^2+3n+1\right)}}\\&=\frac{1}{\sqrt[3]{n^2}}\sqrt[3]{-3n+1+3\sqrt[3]{n(9n^2+3n+1)}}.\end{aligned}$$

Since it is an RCP, using (4.54)*, the cubic Shevelev sum is*

$$\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}}=-\sqrt[3]{\frac{9n^2+3n+1}{n^2}}.$$

4.7.4 RCP2

In this part, we give an example on Ramanujan cubic polynomials of the second kind.

Example 4.7.18. *Let* $f(x)=x^3-\frac{3}{\sqrt[3]{2}}x^2-\frac{3}{\sqrt[3]{2}}x+1=0$. *Denote* $A=-\frac{3}{\sqrt[3]{2}}$, $B=-\frac{3}{\sqrt[3]{2}}$, *and*

$C = 1$. *To show that it is an RCP2:*

$$A^3C + B^3 + 27C^2 = -\frac{27}{2} - \frac{27}{2} + 27 = 0.$$

By using the rational roots theorem, we have $\alpha = -1$ *as a root of* $f(x) = 0$. *Dividing* $f(x)$ *by* $x+1$, *we obtain*

$$2x^2 - \left(2 + 3\sqrt[3]{4}\right)x + 2 = 0.$$

The remaining two roots are

$$\beta = \frac{1}{4}\left(2 + 3\sqrt[3]{4} + \sqrt{\left(2 + 3\sqrt[3]{4}\right)^2 - 16}\right) = \frac{1}{4}\left(2 + 3\sqrt[3]{4} + 2 + 4\sqrt[3]{2} - \sqrt[3]{4}\right) = 1 + \frac{1}{\sqrt[3]{2}} + \sqrt[3]{2},$$
$$\gamma = \frac{1}{4}\left(2 + 3\sqrt[3]{4} - \sqrt{\left(2 + 3\sqrt[3]{4}\right)^2 - 16}\right) = \frac{1}{4}\left(2 + 3\sqrt[3]{4} - 2 - 4\sqrt[3]{2} + \sqrt[3]{4}\right) = \sqrt[3]{4} - \sqrt[3]{2}.$$

Since it is an RCP2, by (4.47), *the root* t *of the associated Ramanujan is*

$$\begin{aligned} t &= \sqrt[3]{\frac{1}{C}\left(A + 3\sqrt[3]{C}\right)\left(B + 3\sqrt[3]{C^2}\right)} + \sqrt[3]{\frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C} + B\right)} \\ &= \sqrt[3]{\left(-\frac{3}{\sqrt[3]{2}} + 3\right)\left(-\frac{3}{\sqrt[3]{2}} + 3\right)} + \sqrt[3]{3\left(-\frac{3}{\sqrt[3]{2}} - \frac{3}{\sqrt[3]{2}}\right)} \\ &= \sqrt[3]{\left(3 - \frac{3}{\sqrt[3]{2}}\right)^2} - \sqrt[3]{9}\sqrt[9]{4}. \end{aligned}$$

By (4.48), *the first Ramanujan-type identity is*

$$-1 + \sqrt[3]{1 + \frac{1}{\sqrt[3]{2}} + \sqrt[3]{2}} + \sqrt[3]{\sqrt[3]{4} - \sqrt[3]{2}} = -\sqrt[3]{-\frac{3}{\sqrt[3]{2}} + 6 + 3\left(\sqrt[3]{\left(3 - \frac{3}{\sqrt[3]{2}}\right)^2} - \sqrt[3]{9}\sqrt[9]{4}\right)}. \qquad (4.72)$$

By (4.49), *the second Ramanujan-type identity is*

$$-1 + \frac{1}{\sqrt[3]{1 + \frac{1}{\sqrt[3]{2}} + \sqrt[3]{2}}} + \frac{1}{\sqrt[3]{\sqrt[3]{4} - \sqrt[3]{2}}} = -\sqrt[3]{-\frac{3}{\sqrt[3]{2}} + 6 + 3\left(\sqrt[3]{\left(3 - \frac{3}{\sqrt[3]{2}}\right)^2} - \sqrt[3]{9}\sqrt[9]{4}\right)}. \qquad (4.73)$$

A byproduct from this example is

$$\sqrt[3]{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}+\sqrt[3]{\sqrt[3]{4}-\sqrt[3]{2}}=\frac{1}{\sqrt[3]{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}}+\frac{1}{\sqrt[3]{\sqrt[3]{4}-\sqrt[3]{2}}}. \tag{4.74}$$

The cubic Shevelev sum is

$$\begin{aligned}&-\sqrt[3]{\frac{1}{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}}-\sqrt[3]{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}+\sqrt[3]{\frac{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}{\sqrt[3]{4}-\sqrt[3]{2}}}\\&+\sqrt[3]{\frac{\sqrt[3]{4}-\sqrt[3]{2}}{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}}-\sqrt[3]{\frac{1}{\sqrt[3]{4}-\sqrt[3]{2}}}-\sqrt[3]{\sqrt[3]{4}-\sqrt[3]{2}}=\sqrt[3]{\left(3-\frac{3}{\sqrt[3]{2}}\right)^2}-\sqrt[3]{9}\sqrt[9]{4}.\end{aligned} \tag{4.75}$$

4.7.5 General Case

In this part, we discuss the cubic equations that are neither an RCP nor RCP2.

Example 4.7.19. *Let $\alpha=1$, $\beta=8$, and $\gamma=27$. Then*

$$A=\alpha+\beta+\gamma=36,\ \alpha\beta+\beta\gamma+\gamma\alpha=8+216+27=251,\ \alpha\beta\gamma=216.$$

The cubic equation with these roots is

$$f(x)=x^3-36x^2+251x-216=0.$$

Denote $A=-36$, $B=251$, and $C=-216$. It is not an RCP since

$$A\sqrt[3]{C}+B+3\sqrt[3]{C^2}=36\cdot 6+251+36=503\neq 0.$$

It neither an RCP2 since

$$A^3C+B^3+27C^2=36^3\cdot 216+251^3+27\cdot 216^2\neq 0.$$

The associated Ramanujan equation of $f(x)$ *is*

$$t^3 - 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right)t - \left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = 0.$$

Now,

$$-3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right) = -3\left(6 + \frac{251}{36} + 3\right) = -\frac{575}{12},$$
$$-\left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = -\left(\frac{251}{6} + 6\left(6 + \frac{251}{36}\right) + 9\right) = -\frac{386}{3}.$$

Then the associated Ramanujan equation is $t^3 - \frac{575}{12}t - \frac{386}{3} = 0.$

By (4.22),

$$\frac{D(f)}{C^2} = \frac{1}{216^2}\left((36 \cdot 251)^2 - 4\left(36^3 \cdot 216 + 251^3\right) + 18 \cdot 36 \cdot 251 \cdot 216 - 27 \cdot 216^2\right)$$

Hence

$$\frac{\sqrt{D(f)}}{C} = \frac{1729}{108}.$$

By (4.21), *the real root* t *is*

$$t = \sqrt[3]{\frac{\frac{386}{3} + \frac{1729}{108}}{2}} + \sqrt[3]{\frac{\frac{386}{3} - \frac{1729}{108}}{2}} = \sqrt[3]{\frac{15625}{216}} + \sqrt[3]{\frac{12167}{216}} = \frac{25}{6} + \frac{23}{6} = 8.$$

Hence by (4.23),

$$\sqrt[3]{1} + \sqrt[3]{8} + \sqrt[3]{27} = 1 + 2 = 3 = 6,$$
$$-\sqrt[3]{-36 - 6 \cdot 6 - 3 \cdot 6 \cdot 8} = 6.$$

On the other hand, by (4.24),

$$\frac{1}{\sqrt[3]{1}} + \frac{1}{\sqrt[3]{8}} + \frac{1}{\sqrt[3]{27}} = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6},$$

$$\frac{1}{6}\sqrt[3]{251+6\cdot 36+3\cdot 8\cdot 36}=\frac{11}{6}.$$

By (4.51)*, the cubic Shevelev sum is*

$$\sqrt[3]{\frac{1}{8}}+\sqrt[3]{\frac{8}{1}}+\sqrt[3]{\frac{1}{27}}+\sqrt[3]{\frac{27}{1}}+\sqrt[3]{\frac{27}{8}}+\sqrt[3]{\frac{8}{27}}=\frac{1}{2}+2+\frac{1}{3}+3+\frac{3}{2}+\frac{2}{3}=8.$$

Recall that $t=8$*. Thus, the cubic Shevelev sum formula holds.*

Example 4.7.20. *Let* $f(x)=x^3-3x^2-6x+18=0$*. Denote* $A=-3$*,* $B=-6$*, and* $C=18$*. It is not an RCP since*

$$A\sqrt[3]{C}+B+3\sqrt[3]{C^2}=-3\sqrt[3]{18}-6+3\sqrt[3]{18^2}\neq 0.$$

It is neither an RCP2 since

$$A^3C+B^3+27C^2=-27(18)-216+27(18^2)=8046\neq 0.$$

To find the roots, α*,* β*, and* γ*, of* $f(x)$*, we first depress it by substituting* $x=y+1$ *to obtain*

$$g(y)=y^3-9y+10=0.$$

The roots of $g(y)$ *are* $\alpha_y=2$*,* $\beta_y=-1+\sqrt{6}$*, and* $\gamma_y=-1-\sqrt{6}$*. Then the roots of* $f(x)$ *are* $\alpha=3$*,* $\beta=\sqrt{6}$*, and* $\gamma=-\sqrt{6}$*. Then the Ramanujan-type identities are*

$$\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}=\sqrt[3]{3}+\sqrt[3]{\sqrt{6}}+\sqrt[3]{-\sqrt{6}}=\sqrt[3]{3}$$

and

$$\frac{1}{\sqrt[3]{\alpha}}+\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}}=\frac{1}{\sqrt[3]{3}}+\frac{1}{\sqrt[3]{\sqrt{6}}}+\frac{1}{\sqrt[3]{-\sqrt{6}}}=\frac{1}{\sqrt[3]{3}}.$$

The associated Ramanujan equation of $f(x)$ is

$$t^3 - 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right)t - \left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = 0.$$

Now,

$$-3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right) = -3\left(\frac{-3}{\sqrt[3]{18}} + \frac{-6}{\sqrt[3]{18^2}} + 3\right) = -3\left(3 - \sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{3}{2}}\right),$$

$$-\left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = -\left(10 + 6\left(-\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{3}{2}}\right)\right).$$

Then the associated Ramanujan equation is

$$t^3 - 3(3+K)t - (6K+10) = 0,$$

where $K = -\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{3}{2}}$. *By* (4.22), *the discriminant is*

$$\frac{D(f)}{C^2} = \frac{1}{18^2}\left(9 \cdot 36 - 4\left(-27(18) - 216\right) + 18(-3)(-6)(18) - 27(18^2)\right) = \frac{2}{3}.$$

By (4.21), *the only real root* t *is*

$$t = \sqrt[3]{\frac{10 + 6\left(-\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{3}{2}}\right) + \sqrt{\frac{2}{3}}}{2}} + \sqrt[3]{\frac{10 + 6\left(-\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{3}{2}}\right) - \sqrt{\frac{2}{3}}}{2}}.$$

We can simplify this quantity by manipulating the associated Ramanujan equation

$$\begin{aligned}
0 &= t^3 - 3(3+K)t - (6K+10) \\
&= t^3 - (3K+9)t - 2(3K+5) \\
&= t^3 - (3K+5)t - 4t - 2(3K+5) \\
&= (t^3 - 4t) - (t+2)(3K+5) \\
&= t(t^2 - 4) - (t+2)(3K+5) \\
&= t(t+2)(t-2) - (t+2)(3K+5)
\end{aligned}$$

$$= (t+2)\left[t(t-2) - 3K + 5\right].$$

So $t = -2$ is the real root. Since by Theorem 4.3.7, there is only one real root t, we can conclude that

$$\sqrt[3]{\frac{10 + 6\left(-\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{3}{2}}\right) + \sqrt{\frac{2}{3}}}{2}} + \sqrt[3]{\frac{10 + 6\left(-\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{3}{2}}\right) - \sqrt{\frac{2}{3}}}{2}} = -2.$$

Thus, the Ramanujan-type identities are

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = -\sqrt[3]{A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}} = -\sqrt[3]{-3 + 6\sqrt[3]{18} - 6\sqrt[3]{18}} = \sqrt[3]{3}$$

and

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = -\frac{1}{\sqrt[3]{C}}\sqrt[3]{B + 6\sqrt[3]{C^2} + 3t\sqrt[3]{C^2}} = -\frac{1}{\sqrt[3]{18}}\sqrt[3]{-6 + 6\sqrt[3]{18^2} - 6\sqrt[3]{18^2}} = \frac{1}{\sqrt[3]{3}}.$$

Example 4.7.21. *(Liao et al. [50]). Let $f(x) = x^3 - 48x - 64\sqrt{2} = 0$. Denote $A = 0$, $B = -48$, and $C = -64\sqrt{2}$. It is not an RCP since*

$$A\sqrt[3]{C} + B + 3\sqrt[3]{C^2} = -48 + 48\sqrt[3]{2} \neq 0.$$

It is also not an RCP2 since

$$A^3C + B^3 + 27C^2 = -48^3 + 27 \cdot 64^2 \cdot 2 \neq 0.$$

The associated Ramanujan equation of $f(x)$ is

$$t^3 + \left(\frac{9}{\sqrt[3]{2}} - 9\right)t + \frac{18}{\sqrt[3]{2}} - 9 = 0.$$

By (4.22),

$$\frac{D(f)}{C^2} = \frac{1}{2 \cdot 64^2}\left(-4 \cdot (-48)^3 - 27 \cdot 2 \cdot 64^2\right) = 27.$$

Hence,

$$\frac{\sqrt{D(f)}}{C} = 3\sqrt{3}.$$

By (4.21)*, the real root t is*

$$t = \sqrt[3]{\frac{9 - \frac{18}{\sqrt[3]{2}} + 3\sqrt{3}}{2}} + \sqrt[3]{\frac{9 - \frac{18}{\sqrt[3]{2}} - 3\sqrt{3}}{2}} = \sqrt[3]{\frac{3}{2}} \left(\sqrt[3]{3 - \frac{6}{\sqrt[3]{2}} + \sqrt{3}} + \sqrt[3]{3 - \frac{6}{\sqrt[3]{2}} - \sqrt{3}} \right).$$

From Liao et al. [50], the roots of $f(x) = 0$ *are:*

$$\alpha = -4\sqrt{2},\ \beta = 2\sqrt{2} + 2\sqrt{6},\ \gamma = 2\sqrt{2} - 2\sqrt{6}.$$

By (4.23)*, the first Ramanujan-type identity is*

$$\begin{aligned} &\sqrt[3]{-4\sqrt{2}} + \sqrt[3]{2\sqrt{2} + 2\sqrt{6}} + \sqrt[3]{2\sqrt{2} - 2\sqrt{6}} \\ &\qquad = -\sqrt[3]{-6 \cdot 4\sqrt[6]{2} - 3\left(4\sqrt[6]{2}\right) \sqrt[3]{\frac{3}{2}} \left(\sqrt[3]{3 - \frac{6}{\sqrt[3]{2}} + \sqrt{3}} + \sqrt[3]{3 - \frac{6}{\sqrt[3]{2}} - \sqrt{3}} \right)} \\ &\qquad = \sqrt[3]{12}\, \sqrt[18]{2} \sqrt[3]{2 + \sqrt[3]{\frac{3}{2}} \left(\sqrt[3]{3 - \frac{6}{\sqrt[3]{2}} + \sqrt{3}} + \sqrt[3]{3 - \frac{6}{\sqrt[3]{2}} - \sqrt{3}} \right)}. \end{aligned}$$

Hence,

$$-\sqrt[3]{2} + \sqrt[3]{1 + \sqrt{3}} + \sqrt[3]{1 - \sqrt{3}} = \sqrt[3]{6\sqrt[3]{4} + 3\sqrt[3]{6} \left(\sqrt[3]{3 - \frac{6}{\sqrt[3]{2}} + \sqrt{3}} + \sqrt[3]{3 - \frac{6}{\sqrt[3]{2}} - \sqrt{3}} \right)}. \tag{4.76}$$

By (4.24)*, the second Ramanujan-type identity is*

$$\begin{aligned} &\frac{1}{\sqrt[3]{-4\sqrt{2}}} + \frac{1}{\sqrt[3]{2\sqrt{2} + 2\sqrt{6}}} + \frac{1}{\sqrt[3]{2\sqrt{2} - 2\sqrt{6}}} \\ &\qquad = \frac{1}{4\sqrt[6]{2}} \sqrt[3]{-48 + 6 \cdot 16\sqrt[3]{2} + 3\left(16\sqrt[3]{2}\right) \sqrt[3]{\frac{3}{2}} \left(\sqrt[3]{3 - \frac{6}{\sqrt[3]{2}} + \sqrt{3}} + \sqrt[3]{3 - \frac{6}{\sqrt[3]{2}} - \sqrt{3}} \right)} \end{aligned}$$

$$= \frac{2\sqrt[3]{6}}{4\sqrt[6]{2}}\sqrt[3]{-1+2\sqrt[3]{2}+\sqrt[3]{3}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right)}.$$

Hence,

$$-\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{1+\sqrt{3}}}+\frac{1}{\sqrt[3]{1-\sqrt{3}}}=\frac{\sqrt[3]{12}}{2}\sqrt[3]{-1+2\sqrt[3]{2}+\sqrt[3]{3}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right)}. \quad (4.77)$$

By (4.51)*, the cubic Shevelev sum is*

$$\sqrt[3]{\frac{-4\sqrt{2}}{2\sqrt{2}+2\sqrt{6}}}+\sqrt[3]{\frac{2\sqrt{2}+2\sqrt{6}}{-4\sqrt{2}}}+\sqrt[3]{\frac{2\sqrt{2}+2\sqrt{6}}{2\sqrt{2}-2\sqrt{6}}}+\sqrt[3]{\frac{2\sqrt{2}-2\sqrt{6}}{2\sqrt{2}+2\sqrt{6}}}+\sqrt[3]{\frac{-4\sqrt{2}}{2\sqrt{2}-2\sqrt{6}}}+\sqrt[3]{\frac{2\sqrt{2}-2\sqrt{6}}{-4\sqrt{2}}}$$
$$=\sqrt[3]{\frac{3}{2}}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right).$$

Simplification gives

$$\sqrt[3]{\frac{1+\sqrt{3}}{1-\sqrt{3}}}+\sqrt[3]{\frac{1-\sqrt{3}}{1+\sqrt{3}}}-\sqrt[3]{\frac{2}{1+\sqrt{3}}}-\sqrt[3]{\frac{1+\sqrt{3}}{2}}-\sqrt[3]{\frac{2}{1-\sqrt{3}}}-\sqrt[3]{\frac{1-\sqrt{3}}{2}}$$
$$=\sqrt[3]{\frac{3}{2}}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right). \quad (4.78)$$

Example 4.7.22. *Let $f(x) = x^3 - 3x + \sqrt{3} = 0$. Denote $A = 0$, $B = -3$, and $C = \sqrt{3}$. It is not an RCP since*

$$A\sqrt[3]{C} + B + 3\sqrt[3]{C^2} = -3 + 3\sqrt[3]{3} \neq 0.$$

It is also not an RCP2 since

$$A^3C + B^3 + 27C^2 = -27 + 27 \cdot 3 \neq 0.$$

The associated Ramanujan equation of $f(x)$ is $t^3 + pt + q = 0$, with

$$p = -3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right) = -3\left(\frac{-3}{\sqrt[3]{3}} + 3\right) = 3\sqrt[3]{9} - 9,$$

$$q=-\left(\frac{AB}{C}+6\left(\frac{A}{\sqrt[3]{C}}+\frac{B}{\sqrt[3]{C^2}}\right)+9\right)=-\left(\frac{-18}{\sqrt[3]{3}}+9\right)=6\sqrt[3]{9}-9.$$

By (4.22),

$$\frac{D(f)}{C^2}=\frac{1}{3}\left(-4\left(-27\right)-27\cdot 3\right)=\frac{1}{3}(27)=9.$$

Then

$$\frac{\sqrt{D(f)}}{C}=3.$$

By (4.21), *the real root* t *is*

$$t=\sqrt[3]{\frac{-6\sqrt[3]{9}+9+3}{2}}+\sqrt[3]{\frac{-6\sqrt[3]{9}+9-3}{2}}=\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}.$$

From Wituła [95], the roots of $f(x)=0$ *are:*

$$\alpha=2\sin\frac{2\pi}{9},\ \beta=-2\sin\frac{4\pi}{9},\ \gamma=2\sin\frac{8\pi}{9}.$$

By (4.23),

$$\sqrt[3]{2\sin\frac{2\pi}{9}}+\sqrt[3]{-2\sin\frac{4\pi}{9}}+\sqrt[3]{2\sin\frac{8\pi}{9}}=-\sqrt[3]{6\sqrt[6]{3}+3\sqrt[6]{3}\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}.$$

Then the first Ramanujan-type identity is

$$\sqrt[3]{\sin\frac{2\pi}{9}}-\sqrt[3]{\sin\frac{4\pi}{9}}+\sqrt[3]{\sin\frac{8\pi}{9}}=-\frac{1}{\sqrt[3]{2}}\sqrt[3]{6\sqrt[6]{3}+3\sqrt[6]{3}\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}. \tag{4.79}$$

By (4.24),

$$\frac{1}{\sqrt[3]{2\sin\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{-2\sin\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{2\sin\frac{8\pi}{9}}}=-\frac{1}{\sqrt[6]{3}}\sqrt[3]{-3+6\sqrt[3]{3}+3\sqrt[3]{3}\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}.$$

Then the second Ramanujan-type identity is

$$\frac{1}{\sqrt[3]{\sin\frac{2\pi}{9}}} - \frac{1}{\sqrt[3]{\sin\frac{4\pi}{9}}} + \frac{1}{\sqrt[3]{\sin\frac{8\pi}{9}}} = \sqrt[3]{2}\,\sqrt[18]{243}\sqrt[3]{\frac{1}{\sqrt[3]{3}} - 2 - \left(\sqrt[3]{6-3\sqrt[3]{9}} + \sqrt[3]{3-3\sqrt[3]{9}}\right)}. \tag{4.80}$$

These Ramanujan-type identities can also be found in Shiue et al. [81], Wang [90]. By (4.51)*, the cubic Shevelev sum is:*

$$-\sqrt[3]{\frac{\sin\frac{2\pi}{9}}{\sin\frac{4\pi}{9}}} - \sqrt[3]{\frac{\sin\frac{4\pi}{9}}{\sin\frac{2\pi}{9}}} - \sqrt[3]{\frac{\sin\frac{4\pi}{9}}{\sin\frac{8\pi}{9}}} - \sqrt[3]{\frac{\sin\frac{8\pi}{9}}{\sin\frac{4\pi}{9}}} + \sqrt[3]{\frac{\sin\frac{2\pi}{9}}{\sin\frac{8\pi}{9}}} + \sqrt[3]{\frac{\sin\frac{8\pi}{9}}{\sin\frac{2\pi}{9}}} = \sqrt[3]{6-3\sqrt[3]{9}} + \sqrt[3]{3-3\sqrt[3]{9}}. \tag{4.81}$$

Example 4.7.23. *Let* $f(x) = x^3 - 3\sqrt[3]{2}x^2 - 3\sqrt[3]{2}x + 1 = 0$*. Denote* $A = -3\sqrt[3]{2} = B$ *and* $C = 1$*. It is not an RCP since*

$$A\sqrt[3]{C} + B + 3\sqrt[3]{C^2} = -3\sqrt[3]{2} + -3\sqrt[3]{2} + 3 \neq 0.$$

It is not an RCP2 since

$$A^3C + B^3 + 27C^2 = -54 - 54 + 27 \neq 0.$$

The associated Ramanujan equation of $f(x)$ *is* $t^3 + pt + = 0$*, with*

$$\begin{aligned} p &= -3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right) = -3\left(-3\sqrt[3]{2} - 3\sqrt[3]{2} + 3\right) = 18\sqrt[3]{2} - 9, \\ q &= -\left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = -\left(9\sqrt[3]{4} + 6\left(-3\sqrt[3]{2} - 3\sqrt[3]{2}\right) + 9\right) \\ &= 36\sqrt[3]{2} - 9\sqrt[3]{4} - 9. \end{aligned}$$

By (4.22),

$$\frac{D(f)}{C^2} = 162\sqrt[3]{2} - 4\left(-54 - 54\right) + 162\sqrt[3]{4} - 27 = 162\sqrt[3]{4} + 162\sqrt[3]{2} + 405.$$

Hence,

$$\sqrt{D(f)} = 9\sqrt{5 + 2\sqrt[3]{2} + 2\sqrt[3]{4}}.$$

By (4.21)*, the real root* t *is*

$$t = \sqrt[3]{\frac{9 + 9\sqrt[3]{4} - 36\sqrt[3]{2} + 9\sqrt{5 + 2\sqrt[3]{2} + 2\sqrt[3]{4}}}{2}} + \sqrt[3]{\frac{9 + 9\sqrt[3]{4} - 36\sqrt[3]{2} - 9\sqrt{5 + 2\sqrt[3]{2} + 2\sqrt[3]{4}}}{2}}.$$

To find the roots of $f(x) = 0$*, we first use the rational roots theorem to obtain one root* $\alpha = -1$*. Divide* $f(x)$ *by* $x + 1$ *yields*

$$x^2 - \left(1 + 3\sqrt[3]{2}\right) x + 1 = 0.$$

The remaining two roots are

$$\beta = \frac{1}{2}\left(1 + 3\sqrt[3]{2} + \sqrt{\left(1 + 3\sqrt[3]{2}\right)^2 - 4}\right) = \frac{1}{2}\left(1 + 3\sqrt[3]{2} + \sqrt{6\sqrt[3]{2} + 9\sqrt[3]{4} - 3}\right),$$

$$\gamma = \frac{1}{2}\left(1 + 3\sqrt[3]{2} - \sqrt{6\sqrt[3]{2} + 9\sqrt[3]{4} - 3}\right).$$

By (4.23)*, the first Ramanujan-type identity is*

$$\begin{aligned} &-1 + \sqrt[3]{\frac{1}{2}\left(1 + 3\sqrt[3]{2} + \sqrt{6\sqrt[3]{2} + 9\sqrt[3]{4} - 3}\right)} + \sqrt[3]{\frac{1}{2}\left(1 + 3\sqrt[3]{2} - \sqrt{6\sqrt[3]{2} + 9\sqrt[3]{4} - 3}\right)} \\ &= \sqrt[3]{3\sqrt[3]{2} - 6 - 3\left(\sqrt[3]{\frac{9 + 9\sqrt[3]{4} - 36\sqrt[3]{2} + 9\sqrt{5 + 2\sqrt[3]{2} + 2\sqrt[3]{4}}}{2}} + \sqrt[3]{\frac{9 + 9\sqrt[3]{4} - 36\sqrt[3]{2} - 9\sqrt{5 + 2\sqrt[3]{2} + 2\sqrt[3]{4}}}{2}}\right)}. \end{aligned} \tag{4.82}$$

By (4.24)*, the second Ramanujan-type identity is*

$$\begin{aligned} &-1 + \frac{1}{\sqrt[3]{\frac{1}{2}\left(1 + 3\sqrt[3]{2} + \sqrt{6\sqrt[3]{2} + 9\sqrt[3]{4} - 3}\right)}} + \frac{1}{\sqrt[3]{\frac{1}{2}\left(1 + 3\sqrt[3]{2} - \sqrt{6\sqrt[3]{2} + 9\sqrt[3]{4} - 3}\right)}} \\ &= \sqrt[3]{3\sqrt[3]{2} - 6 - 3\left(\sqrt[3]{\frac{9 + 9\sqrt[3]{4} - 36\sqrt[3]{2} + 9\sqrt{5 + 2\sqrt[3]{2} + 2\sqrt[3]{4}}}{2}} + \sqrt[3]{\frac{9 + 9\sqrt[3]{4} - 36\sqrt[3]{2} - 9\sqrt{5 + 2\sqrt[3]{2} + 2\sqrt[3]{4}}}{2}}\right)}. \end{aligned} \tag{4.83}$$

A byproduct of this example is

$$\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}+\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}$$
$$=\frac{1}{\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}}+\frac{1}{\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}}. \quad (4.84)$$

By (4.51)*, the cubic Shevelev sum is*

$$-\sqrt[3]{\frac{2}{1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}}}-\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}$$
$$-\sqrt[3]{\frac{2}{1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}}}-\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}$$
$$+\sqrt[3]{\frac{1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}}{1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}}}+\sqrt[3]{\frac{1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}}{1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}}}$$
$$=\sqrt[3]{\frac{9+9\sqrt[3]{4}-36\sqrt[3]{2}+9\sqrt{5+2\sqrt[3]{2}+2\sqrt[3]{4}}}{2}}+\sqrt[3]{\frac{9+9\sqrt[3]{4}-36\sqrt[3]{2}-9\sqrt{5+2\sqrt[3]{2}+2\sqrt[3]{4}}}{2}}. \quad (4.85)$$

Example 4.7.24. *Let* $f(x)=x^3+mx^2+mx+1=0$*, where* $m\in\mathbb{R}$ *and* $m>3$ *or* $m<-1$*. Denote* $A=B=m$ *and* $C=1$*. This is an RCP if* $m=-\frac{3}{2}$*, since* $A\sqrt[3]{C}+B+3\sqrt[3]{C^2}=-\frac{3}{2}-\frac{3}{2}+3=0$*. This is an RCP2 if* $m=-\frac{3}{\sqrt[3]{2}}$*, since* $A^3+B^3+27C^2=-\frac{27}{2}-\frac{27}{2}+27=0$*.*

For the cases of $m\neq-\frac{3}{2}$ *and* $m\neq-\frac{3}{\sqrt[3]{2}}$*,* $f(x)=0$ *is neither an RCP nor RCP2. Note that* $f(x)=0$ *will always have a root* $\alpha=-1$*. The quadratic factor of* $f(x)$ *is* $x^2+(m-1)x+1$*. Then the remaining two roots are*

$$\beta=\frac{1}{2}\left(1-m+\sqrt{m^2-2m-3}\right),$$
$$\gamma=\frac{1}{2}\left(1-m-\sqrt{m^2-2m-3}\right),$$

which are distinct.

The associated Ramanujan equation of $f(x)$ *is* $t^3+pt+q=0$*, with*

$$p=-3\left(\frac{A}{\sqrt[3]{C}}+\frac{B}{\sqrt[3]{C^2}}+3\right)=-3(2m+3)=-(6m+9),$$

$$q = -\left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = -(m^2 + 6(2m) + 9) = -(m^2 + 12m + 9).$$

By (4.22), the discriminant of $f(x)$ is

$$D(f) = m^4 - 4(m^3 + m^3) + 18m^2 - 27 = m^4 - 8m^3 + 18m^2 - 27 = (m-3)^3(m+1).$$

Hence,

$$\sqrt{D(f)} = \sqrt{(m-3)^3(m+1)},$$

since $m > 3$ or $m < -1$. By (4.21), the real root t is

$$t = \sqrt[3]{\frac{m^2 + 12m + 9 + \sqrt{(m-3)^3(m+1)}}{2}} + \sqrt[3]{\frac{m^2 + 12m + 9 - \sqrt{(m-3)^3(m+1)}}{2}}.$$

By (4.23), the first Ramanujan-type identity is

$$-1 + \sqrt[3]{\frac{1}{2}\left(1 - m + \sqrt{m^2 - 2m - 3}\right)} + \sqrt[3]{\frac{1}{2}\left(1 - m - \sqrt{m^2 - 2m - 3}\right)}$$
$$= -\sqrt[3]{m + 6 + 3\left(\sqrt[3]{\frac{m^2 + 12m + 9 + \sqrt{(m-3)^3(m+1)}}{2}} + \sqrt[3]{\frac{m^2 + 12m + 9 - \sqrt{(m-3)^3(m+1)}}{2}}\right)}. \tag{4.86}$$

By (4.24), the second Ramanujan-type identity is

$$-1 + \frac{1}{\sqrt[3]{\frac{1}{2}\left(1 - m + \sqrt{m^2 - 2m - 3}\right)}} + \frac{1}{\sqrt[3]{\frac{1}{2}\left(1 - m - \sqrt{m^2 - 2m - 3}\right)}}$$
$$= -\sqrt[3]{m + 6 + 3\left(\sqrt[3]{\frac{m^2 + 12m + 9 + \sqrt{(m-3)^3(m+1)}}{2}} + \sqrt[3]{\frac{m^2 + 12m + 9 - \sqrt{(m-3)^3(m+1)}}{2}}\right)}. \tag{4.87}$$

A byproduct from this example is

$$\sqrt[3]{\frac{1}{2}\left(1 - m + \sqrt{m^2 - 2m - 3}\right)} + \sqrt[3]{\frac{1}{2}\left(1 - m - \sqrt{m^2 - 2m - 3}\right)}$$
$$= \frac{1}{\sqrt[3]{\frac{1}{2}\left(1 - m + \sqrt{m^2 - 2m - 3}\right)}} + \frac{1}{\sqrt[3]{\frac{1}{2}\left(1 - m - \sqrt{m^2 - 2m - 3}\right)}}. \tag{4.88}$$

Recall β and γ are two roots of the quadratic equation $x^2 + (m-1)x + 1 = 0$. This gives $\beta\gamma = 1$.

Hence, the byproduct is a result of

$$\sqrt[3]{\beta}+\sqrt[3]{\gamma}=\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}}.$$

We also have $\sqrt[3]{\beta}+\frac{1}{\sqrt[3]{\beta}}=\sqrt[3]{\gamma}+\frac{1}{\sqrt[3]{\gamma}}$. *Hence, the Ramanujan-type identities are exactly the same. In addition, the left hand side of the cubic Shevelev sum* (4.51) *can be simplified to*

$$\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\alpha}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\beta}}=-\sqrt[3]{\gamma}-\sqrt[3]{\beta}-\sqrt[3]{\beta}-\sqrt[3]{\gamma}+\sqrt[3]{\beta^2}+\sqrt[3]{\gamma^2}$$
$$=\sqrt[3]{\beta^2}+\sqrt[3]{\gamma^2}-2\sqrt[3]{\beta}-2\sqrt[3]{\gamma}.$$

Then

$$\sqrt[3]{\beta^2}+\sqrt[3]{\gamma^2}-2\sqrt[3]{\beta}-2\sqrt[3]{\gamma}=t.$$

Thus, the cubic Shevelev sum becomes

$$\sqrt[3]{\frac{\left(1-m+\sqrt{m^2-2m-3}\right)^2}{4}}+\sqrt[3]{\frac{\left(1-m-\sqrt{m^2-2m-3}\right)^2}{4}}$$
$$-2\sqrt[3]{\frac{1-m+\sqrt{m^2-2m-3}}{2}}-2\sqrt[3]{\frac{1-m-\sqrt{m^2-2m-3}}{2}}$$
$$=\sqrt[3]{\frac{m^2+12m+9+\sqrt{(m-3)^3(m+1)}}{2}}+\sqrt[3]{\frac{m^2+12m+9-\sqrt{(m-3)^3(m+1)}}{2}}. \tag{4.89}$$

4.7.6 Constructing Cosine Ramanujan-type Identities

To construct cosine Ramanujan-type identities by using Theorem 4.3.9, we choose a suitable $r\in\mathbb{C}$. For a more general result, we can use Corollary 4.3.10 and choose any $r\in\mathbb{C}$ and $A\in\mathbb{R}$.

Dresden et al. [22] gave an example similar to the following example. We use our approach in the following example.

Example 4.7.25. *Let* $r=\frac{\sqrt{3}}{2}+\frac{1}{2}i$. *Then* $s=\frac{\sqrt{3}}{2}-\frac{1}{2}i$, $r+s=\sqrt{3}$, $rs=|r|^2=1$, *and*

$\theta = \text{Arg}(r) = \dfrac{\pi}{6}$. *By Theorem 4.3.9, the cubic equation is* $x^3 - 3x + \sqrt{3} = 0$. *The roots are*

$$\alpha = -2\sqrt{rs}\cos\left(\frac{\theta}{3}\right),\ \beta = -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right),\ \gamma = -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right).$$

Then

$$\alpha = -2\cos\frac{\pi}{18},\ \beta = -2\cos\frac{13\pi}{18},\ \gamma = -2\cos\frac{25\pi}{18}.$$

The real root t *to the associated Ramanujan equation is*

$$\begin{aligned} t &= \sqrt[3]{\frac{9\left(1 - \sqrt[3]{\frac{2}{\xi^2}}\right) + 3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}} + \sqrt[3]{\frac{9\left(1 - \sqrt[3]{\frac{2}{\xi^2}}\right) - 3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}} \\ &= \sqrt[3]{\frac{9\left(1 - 2\sqrt[3]{\frac{1}{3}}\right) + 3\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)}{2}} + \sqrt[3]{\frac{9\left(1 - 2\sqrt[3]{\frac{1}{3}}\right) - 3\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)}{2}} \\ &= \sqrt[3]{6 - \frac{9}{\sqrt[3]{3}}} + \sqrt[3]{3 - \frac{9}{\sqrt[3]{3}}} = \sqrt[3]{6 - 3\sqrt[3]{9}} + \sqrt[3]{3 - 3\sqrt[3]{9}} \end{aligned}$$

By (4.35),

$$\sqrt[3]{\cos\left(\frac{\theta}{3}\right)} + \sqrt[3]{\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)} + \sqrt[3]{\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right)} = \sqrt[9]{2\xi}\sqrt[3]{\frac{6 + 3t}{2}}.$$

Then the first Ramanujan-type identity is

$$\sqrt[3]{\cos\frac{\pi}{18}} + \sqrt[3]{\cos\frac{13\pi}{18}} + \sqrt[3]{\cos\frac{25\pi}{18}} = \frac{\sqrt[18]{3}}{\sqrt[3]{2}}\sqrt[3]{6 + 3\left(\sqrt[3]{6 - 3\sqrt[3]{9}} + \sqrt[3]{3 - 3\sqrt[3]{9}}\right)}$$

By (4.36),

$$\frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}\right)}} + \frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)}} + \frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right)}} = \sqrt[3]{\frac{3}{\xi}}\sqrt[3]{-1 + 2\sqrt[3]{4\xi^2} + t\sqrt[3]{4\xi^2}}.$$

Then the second Ramanujan-type identity is

$$\frac{1}{\sqrt[3]{\cos\frac{\pi}{18}}} + \frac{1}{\sqrt[3]{\cos\frac{13\pi}{18}}} + \frac{1}{\sqrt[3]{\cos\frac{25\pi}{18}}} = \sqrt[3]{2}\sqrt[6]{3}\sqrt[3]{-1 + 2\sqrt[3]{3} + \sqrt[3]{3}\left(\sqrt[3]{6 - 3\sqrt[3]{9}} + \sqrt[3]{3 - 3\sqrt[3]{9}}\right)}.$$

4.8 Conclusion

In this chapter, we have made significant contributions towards simplifying the computational complexity involved in obtaining Wang's Ramanujan-type identities. By introducing a novel approach based on Liao et al.'s method applied to the associated Ramanujan equation of cubic equations, we have achieved substantial improvements in computational efficiency. Our method, relying solely on arithmetic calculations, markedly reduces the total amount of computations required compared to Wang's original method.

Furthermore, we have established connections between Ramanujan, Shevelev, and Cardano cubic equations, culminating in the development of a generalized computation procedure. This procedure enables the construction of various related identities, including cosine Ramanujan-type identities. By bridging sixteenth-century insights with contemporary numerical treatments, our work contributes to the ongoing discourse in understanding cubic equations and their interrelated numerical treatments.

This chapter not only sheds light on historical advancements but also opens avenues for practical applications in fields reliant on efficient computational methods. Moreover, it underscores the enduring relevance of Ramanujan's work and its implications for modern mathematical inquiry, echoing the ongoing dialogue initiated by Newton's pioneering efforts in understanding non-linear functions.

The publications can be found in Shiue et al. [82] and Shiue et al. [83].

CHAPTER 5

ON INFINITE SERIES SUMMATIONS INVOLVING LINEAR RECURRENCE RELATIONS ORDER 2 AND 3

5.1 Background

In recent years, a renewed interest in Leonardo and Fibonacci sequences has emerged, partly ignited by the research of Catarino and Borges [11]. This revival echoes early discussions within the Fibonacci Association (Bicknell-Johnson and Bergum [8]), which sought to underscore the intellectual legacy of Fibonacci. Historically, the focus has largely been on transforming non-homogeneous second-order equations into higher-order homogeneous forms, which may have led to a somewhat restricted investigation of the number-theoretic characteristics unique to Leonardo sequences. Expanding upon the foundational work of Horadam [33], this chapter delves into the domain of non-homogeneous properties within the Leonardo framework, as evidenced in subsequent tables that beckon further inquiry. The application of this framework intersects with well-established sequences from Koshy [44], culminating in a collection of identities and combinatorial insights associated with generalized Leonardo sequences. This exploration offers fertile ground for continued mathematical investigation.

In parallel, this chapter focuses on infinite series derived from sequences that generalize both the Fibonacci sequences and the so-called Tribonacci sequences. This research is anchored in seminal contributions made by Melham and Shannon [58] three decades ago, as reflected in Table 1, which succinctly summarizes key findings for readers to explore further. The core theorems of this study revolve around the roots of quadratic characteristic equations governing second-order sequences, elucidating conditions for convergence in resulting infinite series. Significantly, these theorems integrate insights from prominent mathematicians such as Henry Gould, Rudi Lidl, Harald Niederreiter, and Morgan Ward, establishing vital connections to prior research.

$\{a_{mn}\}$	$m=1$	$m=2$	$m=3$
$\{M_{mn}\}$	1/72	3/54	7/18
$\{S_{mn}\}$	17/72	15/54	11/18
$\{P_{mn}\}$	1/79	2/41	#
$\{Q_{mn}\}$	18/79	14/41	#
$\{J_{mn}\}$	1/88	1/54	3/22
$\{j_{mn}\}$	19/88	15/54	13/22
$\{F_{mn}\}$	1/89	1/71	2/59
$\{F_{mn+1}\}$	10/89	9/71	9/59
$\{L_{mn}\}$	19/89	17/71	16/59
$\{Le_{mn}\}$	91/801	91/639	103/531
$\{\mathcal{L}_{mn}\}$	171/801	153/639	144/531

Table 5.1: $\sum_{n=0}^{\infty} \frac{a_{mn}}{10^{n+1}}$, $m = 1, 2, 3$. # Does not satisfy the condition for convergence. The special sequences are defined in Table 5.2. Le_n and $\mathcal{L}_n$ are defined in Section 5.4.4.

In this chapter, we embark on a comprehensive exploration split into two parts, each delving into fundamental aspects of sequences and series.

Firstly, we direct our attention to analyzing the sequence $w_n(w_0, w_1, p, q, t, j)$ of order 2, which adheres to a non-homogeneous linear relation of the form:

$$w_n = pw_{n-1} + qw_{n-2} + (p + q - 1)(tn + j), \; n \geq 2, \tag{5.1}$$

where $w_0, \; w_1, \; p,$ and q are predetermined constants, with $p + q \neq 1$, while t and j are integer parameters.

Following this analysis, our focus shifts towards a systematic examination of series characterized by the expression:

$$\sum_{n=0}^{\infty} \frac{a_{mn}}{10^{n+1}}$$

Here, we endeavor to elucidate the essence of these series and provide insightful interpretations of their values. Through rigorous computation and analysis, we unveil the numerical outcomes of these series, offering a glimpse into their convergence properties and potential implications across mathematical domains.

By meticulously dissecting these two facets—sequences governed by non-homogeneous linear relations and the convergence behavior of series—we aim to enrich our understanding of fundamental mathematical structures while paving the way for novel insights and discoveries in the realm of mathematical analysis.

5.2 Preliminaries

Definition 5.2.1. *A number sequence $\{a_n\}$ is called a sequence of order 2 if it satisfies the recurrence relation of order 2:*

$$a_n = pa_{n-1} + qa_{n-2}, \ n \geq 2, \tag{5.2}$$

for some constants $p, q \neq 0$ and initial conditions a_0, a_1.

Given the initial conditions a_0, a_1, and the recurrence relation (5.2), the entire sequence can be determined. Many common sequences, such as arithmetic sequences, geometric sequences, and Fibonacci sequences, are examples of second order linear recurrence sequence (refer to table below, see Horadam [32]).

The purpose of this research is to investigate the series in the form of $\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}$, with $r > 1$. One well-known series of this form, which piqued the curiosity of many and inspired this research, is the Fibonacci sum $\sum_{n=0}^{\infty} \frac{F_n}{10^{n+1}} = \frac{1}{89} = \frac{1}{F_{11}}$ (Long [55]).

Sequence of numbers $\{a_n\}$	a_0	a_1	p	q
Integers $0, 1, 2, 3, \cdots$	0	1	2	-1
Arithmetic sequence (common difference d)	a	$a+d$	2	-1
Geometric sequence (common ratio r)	a	r	$r+1$	$-r$
Fibonacci sequence F_n	0	1	1	1
Lucas sequence L_n	2	1	1	1
Fermat sequence of the first kind T_n	1	3	3	-2
Fermat sequence of the second kind S_n	2	3	3	-2
Pell sequence of the first kind P_n	1	2	2	1
Pell sequence of the second kind Q_n	2	2	2	1
Balancing sequence B_n	0	1	6	-1
Lucas-Balancing sequence C_n	1	3	6	-1
Mersenne sequence M_n	0	1	3	-2
Jacobsthal sequence J_n	0	1	1	2
Jacobsthal-Lucas sequence $\mathcal{J}_n$	2	1	1	2

Table 5.2: Some common sequences.

5.3 Sequence $w_n = pw_{n-1} + qw_{n-2} + (p+q-1)(tn+j)$

We begin by giving some definitions and previously established results.

Definition 5.3.1. *(Kuhapatanakul and Chobsorn [45]). The generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}$, with a fixed positive integer k, is defined by*

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k, \; n \geq 2, \tag{5.3}$$

with the initial conditions $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$.

Furthermore, a variant of generalized Leonardo sequence, $\{B_n(a, b, k)\}$, is defined as follows

Definition 5.3.2. *(Bicknell-Johnson and Bergum [8]). The Leonardo-like sequence $\{B_n(a, b, k)\}$*

is defined by the relation

$$B_n(a,b,k) = B_{n-1}(a,b,k) + B_{n-2}(a,b,k) + k, \tag{5.4}$$

for $n \geq 2$ *and initial values* $B_0(a,b,k) = -a+b-k$ *and* $B_1(a,b,k) = a$. *Here,* k *remains constant.*

The work of Bicknell-Johnson and Bergum [8] showcased that the generalized Leonardo sequence is a specific instance of B_n, where $\mathcal{L}_{k,n} = B_n(1, 2+k, k)$.

Theorem 5.3.1. *(Kuhapatanakul and Chobsorn [45]). The closed formula for the generalized Leonardo sequence* $\{\mathcal{L}_{k,n}\}$ *is*

$$\mathcal{L}_{k,n} = (1+k)F_{n+1} - k, \tag{5.5}$$

Additionally, a corollary by Catarino and Borges [11] discussed the classical Leonardo sequence, Le_n:

Corollary 5.3.2. *(Catarino and Borges [11]). Let* $\{Le_n\}$ *be the classical Leonardo sequence. Then*

$$Le_n = 2F_{n+1} - 1.$$

This result can be derived by setting $k = 1$ in Theorem 5.3.1.

In the next corollary, we present two important relationships within the generalized Leonardo sequence:

Corollary 5.3.3. *Let* $\{\mathcal{L}_{k,n}\}$ *be the generalized Leonardo sequence. Then*

(i) $\mathcal{L}_{k,n} = \frac{1}{2}(1+k)(\mathcal{L}_{1,n} - 1) + 1$,

(ii) $\mathcal{L}_{k+1,n} - \mathcal{L}_{k,n} = \frac{1}{2}(\mathcal{L}_{1,n} - 1)$.

Proof. Referencing Corollary 5.3.2, we establish that $\mathcal{L}_{1,n} = Le_n = 2F_{n+1} - 1$. Consequently, it follows that $F_{n+1} = \frac{1}{2}(\mathcal{L}_{1,n} + 1)$. Employing Theorem 5.3.1, we deduce:

$$\mathcal{L}_{k,n} = (1+k)F_{n+1} - k = \frac{1}{2}(1+k)(\mathcal{L}_{1,n} + 1) - k = \frac{1}{2}(1+k)(\mathcal{L}_{1,n} - 1) + 1,$$

thereby confirming *(i)*.

In a similar vein, as per Theorem 5.3.1, we find:

$$\mathcal{L}_{k+1,n} = (k+2)F_{n+1} - (k+1),$$
$$\mathcal{L}_{k,n} = (k+1)F_{n+1} - k.$$

The subtraction of these two expressions yields:

$$\begin{aligned}\mathcal{L}_{k+1,n} - \mathcal{L}_{k,n} &= (k+2)F_{n+1} - k - 1 - ((k+1)F_{n+1} - k)\\ &= F_{n+1} - 1 = \frac{1}{2}(\mathcal{L}_{1,n} - 1),\end{aligned}$$

which substantiates *(ii)*. □

5.3.1 Main Results

Let $\{a_n\}$ be a sequence of order 2 satisfying the following homogeneous linear recurrence relation:

$$a_n = pa_{n-1} + qa_{n-2}, \quad n \geq 2, \tag{5.6}$$

where $a_0, a_1, p, q \neq 0$ are given constants. Let α and β be two roots of the characteristic equation of (5.2):

$$x^2 - px - q = 0. \tag{5.7}$$

He and Shiue [27] proved the following theorem that gives the general formula of $\{a_n\}$.

Theorem 5.3.4. *(He and Shiue [27]). Let* $\{a_n\}$ *be a sequence of order* 2 *satisfying the linear recurrence relation* (5.2)*. Then*

$$a_n = \begin{cases} \left(\dfrac{a_1 - \beta a_0}{\alpha - \beta}\right)\alpha^n - \left(\dfrac{a_1 - \alpha a_0}{\alpha - \beta}\right)\beta^n, & \text{if } \alpha \neq \beta; \\ na_1\alpha^{n-1} - (n-1)a_0\alpha^n, & \text{if } \alpha = \beta, \end{cases} \tag{5.8}$$

where α and β are the two roots of (5.7).

Corollary 5.3.5. *If $a_0 = 0$ and $a_1 = 1$, then the general formula is given by*

$$a_n = \begin{cases} \left(\dfrac{1}{\alpha - \beta}\right)\alpha^n - \left(\dfrac{1}{\alpha - \beta}\right)\beta^n, & \text{if } \alpha \neq \beta; \\ n\alpha^{n-1}, & \text{if } \alpha = \beta. \end{cases} \tag{5.9}$$

Corollary 5.3.6. *If $a_0 = 1$ and $a_1 = 1$, then the general formula is given by*

$$a_n = \begin{cases} \left(\dfrac{1-\beta}{\alpha - \beta}\right)\alpha^n - \left(\dfrac{1-\alpha}{\alpha - \beta}\right)\beta^n, & \text{if } \alpha \neq \beta; \\ n\alpha^{n-1} - (n-1)\alpha^n, & \text{if } \alpha = \beta. \end{cases} \tag{5.10}$$

Theorem 5.3.4, Corollary 5.3.5, and Corollary 5.3.6 will be used in the main results.

In this paper, we will consider the sequence $\{a_n(t, j)\}$ satisfying the second order non- homogeneous linear recurrence relation:

$$a_n(t, j) = pa_{n-1}(t, j) + qa_{n-2}(t, j) + (p + q - 1)(tn + j), \quad n \geq 2, \ t, j \in \mathbb{Z}, \tag{5.11}$$

where $a_0(t, j)$, $a_1(t, j)$, p, and q, with $p + q \neq 1$, are given constants.

We will write $a_n(t, j)$ as w_n, to follow Horadam [33] notation:

$$w_n \equiv w_n(w_0, w_1, p, q, t, j) = a_n(t, j), \tag{5.12}$$

with $w_0 = a_0(t, j)$, $w_1 = a_1(t, j)$, $w_n(w_0, w_1, p, q, 0, 0) = a_n$, $n \geq 2$.

We now give the general formula of w_n:

Theorem 5.3.7. *Let $\{w_n(w_0, w_1, p, q, t, j)\}$ be a sequence of order 2 satisfying the non-homogeneous linear relation in the following form:*

$$w_n = pw_{n-1} + qw_{n-2} + (p + q - 1)(tn + j), \ n \geq 2, \ t, j \in \mathbb{Z}, \tag{5.13}$$

where $w_0,\ w_1,\ p,\ q,$ *with* $p+q\neq 1,$ *are given constants. Then*

$$w_n = w_n(w_0, w_1, p, q, 0, 0) + \left(j - \frac{t(p+2q)}{1-p-q}\right)(w_n(1, 1, p, q, 0, 0) - 1) + t\left(w_n(0, 1, p, q, 0, 0) - n\right). \tag{5.14}$$

Proof. First we consider the homogeneous part

$$w_n(w_0, w_1, p, q, 0, 0) = pw_{n-1}(w_0, w_1, p, q, 0, 0) + qw_{n-2}(w_0, w_1, p, q, 0, 0).$$

Then the characteristic equation

$$x^2 = px + q$$

gives

$$x = \frac{p \pm \sqrt{p^2+4q}}{2}.$$

Let $\alpha = \dfrac{p+\sqrt{p^2+4q}}{2}$ and $\beta = \dfrac{p-\sqrt{p^2+4q}}{2}$. Then the homogeneous solution of (5.13) is

$$w_n(w_0, w_1, p, q, 0, 0) = c_1\alpha^n + c_2\beta^n.$$

Suppose $\alpha \neq \beta$. Assume the particular solution is of the form

$$w_n^\rho = An + B,$$

where $A = A(t, j)$ and $B = B(t, j)$. Then we have

$$An + B = p(A(n-1) + B) + q(A(n-2) + B) + (p+q-1)(tn + j).$$

Solving for A and B, we have

$$A = -t,$$
$$B = \frac{t(p+2q)}{1-p-q} - j.$$

Then

$$w_n = c_1\alpha^n + c_2\beta^n - tn - j + \frac{t(p+2q)}{1-p-q}.$$

Using the initial conditions w_0 and w_1, we have

$$\begin{cases} w_0 &= c_1 + c_2 - j + \dfrac{t(p+2q)}{1-p-q} \\ w_1 &= c_1\alpha + c_2\beta - t - j + \dfrac{t(p+2q)}{1-p-q}. \end{cases}$$

Multiplying the first equation by α and subtract with the second, we have

$$\begin{aligned} w_0\alpha - w_1 &= c_2(\alpha-\beta) + \left(-j + \frac{t(p+2q)}{1-p-q}\right)\alpha + t + j + \frac{t(p+2q)}{1-p-q} \\ \implies c_2 &= \frac{w_0\alpha - w_1}{\alpha-\beta} - \frac{t}{\alpha-\beta} + \left(-j + \frac{t(p+2q)}{1-p-q}\right)\left(\frac{1-\alpha}{\alpha-\beta}\right). \end{aligned}$$

Then

$$c_1 = \frac{w_1 - w_0\beta}{\alpha-\beta} + \frac{t}{\alpha-\beta} + \left(-j + \frac{t(p+2q)}{1-p-q}\right)\left(\frac{\beta-1}{\alpha-\beta}\right).$$

Thus, the general solution is

$$\begin{aligned} w_n = &\left[\frac{w_1 - w_0\beta}{\alpha-\beta} + \frac{t}{\alpha-\beta} + \left(-j + \frac{t(p+2q)}{1-p-q}\right)\left(\frac{\beta-1}{\alpha-\beta}\right)\right]\alpha^n \\ &+ \left[\frac{w_0\alpha - w_1}{\alpha-\beta} - \frac{t}{\alpha-\beta} + \left(-j + \frac{t(p+2q)}{1-p-q}\right)\left(\frac{1-\alpha}{\alpha-\beta}\right)\right]\beta^n \\ &- tn - j + \frac{t(p+2q)}{1-p-q}. \end{aligned} \tag{5.15}$$

We can rewrite it as

$$w_n = \left(\frac{w_1 - w_0\beta}{\alpha-\beta}\right)\alpha^n - \left(\frac{w_1 - w_0\alpha}{\alpha-\beta}\right)\beta^n - \left(j - \frac{t(p+2q)}{1-p-q}\right)\left(\frac{(\beta-1)\alpha^n + (1-\alpha)\beta^n}{\alpha-\beta}\right)$$
$$+ t\left(\frac{\alpha^n - \beta^n}{\alpha-\beta}\right) - tn - j + \frac{t(p+2q)}{1-p-q}.$$

Using (5.8), (5.9), and (5.10), we have

$$\begin{aligned} w_n = w_n(w_0, w_1, p, q, 0, 0) + \left(j - \frac{t(p+2q)}{1-p-q}\right)(w_n(1,1,p,q,0,0) - 1) \\ + t\left(w_n(0,1,p,q,0,0) - n\right). \end{aligned} \tag{5.16}$$

for $\alpha \neq \beta$.

Now, if $\alpha = \beta$, we have

$$w_n = (c_1 + c_2 n)\,\alpha^n.$$

The solution is

$$w_n = (c_1 + c_2 n)\,\alpha^n - tn - j + \frac{t(p+2q)}{1-p-q}.$$

Using the initial conditions,

$$\begin{aligned} w_0 &= c_1 - j + \frac{t(p+2q)}{1-p-q} \\ w_1 &= c_1\alpha + c_2\alpha - t - j + \frac{t(p+2q)}{1-p-q}. \end{aligned}$$

Then

$$\begin{aligned} c_1 &= w_0 + j - \frac{t(p+2q)}{1-p-q} \\ c_2 &= \frac{w_1}{\alpha} - w_0 - j + \frac{t(p+2q)}{1-p-q} + \frac{1}{\alpha}\left(t + j - \frac{t(p+2q)}{1-p-q}\right) \end{aligned}$$

$$
\begin{aligned}
w_n &= \left(w_0 + j - \frac{t(p+2q)}{1-p-q}\right)\alpha^n + \left[\frac{w_1}{\alpha} - w_0 - j + \frac{t(p+2q)}{1-p-q} + \frac{1}{\alpha}\left(t + j - \frac{t(p+2q)}{1-p-q}\right)\right] n\alpha^n \\
&\quad - tn - j + \frac{t(p+2q)}{1-p-q} \\
&= \left(w_0 + j - \frac{t(p+2q)}{1-p-q}\right)(\alpha^n - n\alpha^n) + \left(w_1 + t + j - \frac{t(p+2q)}{1-p-q}\right) n\alpha^{n-1} \\
&\quad - tn - j + \frac{t(p+2q)}{1-p-q} \\
&= w_1 n\alpha^{n-1} - w_0(n-1)\alpha^n + \left(j - \frac{t(p+2q)}{1-p-q}\right)\big(n\alpha^{n-1} - (n-1)\alpha^n - 1\big) + tn\alpha^{n-1} - tn.
\end{aligned}
$$

Thus, if $\alpha = \beta$, the solution is

$$
w_n = w_1 n\alpha^{n-1} - w_0(n-1)\alpha^n + \left(j - \frac{t(p+2q)}{1-p-q}\right)\big(n\alpha^{n-1} - (n-1)\alpha^n - 1\big) + tn\alpha^{n-1} - tn.
$$

Using (5.8), (5.9), and (5.10), then

$$
\begin{aligned}
w_n = w_n(w_0, w_1, p, q, 0, 0) &+ \left(j - \frac{t(p+2q)}{1-p-q}\right)(w_n(1, 1, p, q, 0, 0) - 1) \\
&+ t(w_n(0, 1, p, q, 0, 0) - n).
\end{aligned}
$$

The results for both cases are the same. □

Corollary 5.3.8. *(Bicknell-Johnson and Bergum [8]). Consider the Leonardo-like sequence* $C_n(a, b, k)$ *defined in* (5.4)*. Using Horadam's notation, we have*

$$
w_n = w_n(b - a - k, a, 1, 1, 0, k).
$$

Then

$$
w_n(b - a - k, a, 1, 1, 0, k) = aF_{n-2} + bF_{n-1} + k(F_n - 1). \tag{5.17}
$$

Proof. By Theorem 5.3.7,

$$\begin{aligned} w_n(b-a-k,a,1,1,0,k) &= w_n(b-a-k,a,1,1,0,0)+k(w_n(1,1,1,1,0,0)-1) \\ &= w_n(b-a-k,a,1,1,0,0)+k(F_{n+1}-1). \end{aligned}$$

Since $p = q = 1$ and $t = j = 0$, we can use Theorem 5.3.4, with $\alpha = \phi$ and $\beta = \psi$:

$$\begin{aligned} w_n(b-a-k,a,1,1,0,0) &= \frac{a-\psi(b-a-k)}{\phi-\psi}\phi^n - \frac{a-\phi(b-a-k)}{\phi-\psi}\psi^n \\ &= a\left(\frac{\phi^n-\psi^n}{\phi-\psi}\right) + (b-a-k)\left(\frac{\phi^{n-1}-\psi^{n-1}}{\phi-\psi}\right) \\ &= aF_n + (b-a-k)F_{n-1} = aF_{n-2} + (b-k)F_{n-1}. \end{aligned}$$

Hence, by Theorem 5.3.7,

$$\begin{aligned} w_n(b-a-k,a,1,1,0,k) &= aF_{n-2} + (b-k)F_{n-1} + k(F_{n+1}-1) \\ &= aF_{n-2} + bF_{n-1} + k(F_n - 1).\square \end{aligned}$$

Corollary 5.3.9. *Consider the general Leonardo sequence* $\{w_n(1,1,1,1,t,j)\}$. *Then*

$$w_n(1,1,1,1,t,j) = (1+3t+j)F_{n+1} + t(F_n - n - 3) - j. \tag{5.18}$$

Proof. Let $p = q = 1$ and $w_0 = w_1 = 1$ in (5.14) of Theorem 5.3.7. Recall that the Fibonacci sequence $\{F_n\}$ satisfies the second order linear recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \tag{5.19}$$

where $F_0 = 0$ and $F_1 = 1$. By (5.9), we have

$$w_n(0,1,1,1,0,0) = F_n,\ w_n(1,1,1,1,0,0) = F_{n+1}$$

where $\alpha = \phi$ and $\beta = \psi$. Then

$$\begin{aligned} w_n(1,1,1,1,t,j) &= w_n(1,1,1,1,0,0) + \left(j - \frac{t(1+2)}{1-1-1}\right)(w_n(1,1,1,1,0,0) - 1) \\ &\quad + t\,(w_n(0,1,1,1,0,0) - n) \\ &= F_{n+1} + (j+3t)\,(F_{n+1} - 1) + t(F_n - n) \\ &= (1+j+3t)F_{n+1} + tF_n - tn - j - 3t \\ &= (1+3t+j)F_{n+1} + t(F_n - n - 3) - j. \square \end{aligned}$$

Corollary 5.3.10. *(Shannon and Deveci [74]). Consider the sequence* $\{w_n(1,1,1,1,1,j)\}$ *of order 2 satisfying the non-homogeneous linear recurrence relation* (5.13). *Then*

$$w_n(1,1,1,1,1,j) = (4+j)F_{n+1} + F_n - n - 3 - j. \tag{5.20}$$

Proof. Let $t = 1$ in Corollary 5.3.9 yield the result. □

Corollary 5.3.11. *The closed formula for the generalized Leonardo sequence* $\{\mathcal{L}_{k,n}\}$ *defined in Definition 5.3.1 is*

$$\mathcal{L}_{k,n} = (1+k)F_{n+1} - k,$$

as given in Theorem 5.3.1.

Proof. Let $t = 0$ and $j = k$ in Corollary 5.3.9 yield the result. □

Corollary 5.3.12. *(Shannon and Deveci [74].) Consider the sequence* $\{w_n(1,1,1,1,1,0)\}$ *of order 2 satisfying the non-homogeneous linear recurrence relation* (5.13). *Then*

$$w_n(1,1,1,1,1,0) = 4F_{n+1} + F_n - n - 3 \tag{5.21}$$

Proof. Let $t = 1$ and $j = 0$ in Corollary 5.3.9 yield the result. □

Theorem 5.3.13. *Let $\{w_n(w_0, w_1, p, q, t, j)\}$ be a sequence of order 2 satisfying the non-homogeneous linear relation in the following form:*

$$w_n = pw_{n-1} + qw_{n-2} + (p+q-1)(tn+j), \;\; n \geq 2, \;\; t, j \in \mathbb{Z}, \tag{5.22}$$

where $w_0, \; w_1, \; p, \; q,$ with $p+q \neq 1$, are given constants. Then

$$w_n(w_0, w_1, p, q, t, j+1) - w_n(w_0, w_1, p, q, t, j) = w_n(1, 1, p, q, 0, 0) - 1 \text{ and} \tag{5.23}$$

$$w_n(w_0, w_1, p, q, t, j+k) - w_n(w_0, w_1, p, q, t, j) = k\left(w_n(1, 1, p, q, 0, 0) - 1\right). \tag{5.24}$$

Proof. Using the result from Theorem 5.3.7, we have

$$w_n(w_0, w_1, p, q, t, j+1) = w_n(w_0, w_1, p, q, 0, 0) + \left(j+1-\frac{t(p+2q)}{1-p-q}\right)(w_n(1, 1, p, q, 0, 0)-1)$$
$$+ t\left(w_n(0, 1, p, q, 0, 0) - n\right)$$

and

$$w_n(w_0, w_1, p, q, t, j) = w_n(w_0, w_1, p, q, 0, 0) + \left(j-\frac{t(p+2q)}{1-p-q}\right)(w_n(1, 1, p, q, 0, 0)-1)$$
$$+ t\left(w_n(0, 1, p, q, 0, 0) - n\right).$$

Subtracting the two equations yields

$$w_n(w_0, w_1, p, q, t, j+1) - w_n(w_0, w_1, p, q, t, j) = w_n(1, 1, p, q, 0, 0) - 1.$$

The second result can be obtained by repeating the same process and replacing $j+1$ by $j+k$. □

Corollary 5.3.14. *Consider the Leonardo-like sequence $\{w_n(b-a-k, a, 1, 1, 0, k)\}$. Then for $n \geq 2$,*

$$w_n(b-a-k, a, 1, 1, 0, k+1) - w_n(b-a-k, a, 1, 1, 0, k) = F_{n+1} - 1.$$

Proof. By Theorem 5.3.13,

$$\begin{aligned} w_n(b-a-k,a,1,1,0,k+1) - w_n(b-a-k,a,1,1,0,k) &= w_n(1,1,1,1,0,0) - 1 \\ &= F_{n+1} - 1. \square \end{aligned}$$

Corollary 5.3.15. *(Shannon [73]). Consider the general Leonardo sequence $\{w_n(1,1,1,1,t,j)\}$. Then for $n \geq 2$,*

$$w_n(1,1,1,1,t,j+1) - w_n(1,1,1,1,t,j) = F_{n+1} - 1. \tag{5.25}$$

Proof. Using Theorem 5.3.13. We have

$$w_n(1,1,1,1,t,j+1) - w_n(1,1,1,1,t,j) = w_n(1,1,1,1,0,0) - 1 = F_{n+1} - 1. \square$$

Note that $w_n(1,1,1,1,0,k) = \mathcal{L}_{k,n}$. Hence when $t = 0$, we have the same result as Corollary 5.3.3 (ii).

Next, note that this difference is independent of t. A table by Shannon and Deveci [74] for $t = 1$ is given here:

j \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
−3	1	1	1	2	4	8	15	27	47
−2	1	1	2	4	8	15	27	47	80
−1	1	1	3	6	12	22	39	67	113
0	1	1	4	8	16	29	51	87	146
1	1	1	5	10	20	36	63	107	179
2	1	1	6	12	24	43	75	127	212
3	1	1	7	14	28	50	87	147	245
Differences	0	0	1	2	4	7	12	20	33

Table 5.3: "Extended Leonardo sequence", Shannon and Deveci [74].

We now give two more tables with $t = 2$ and $t = 3$ to show the Independence of t:

j \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
−3	1	1	3	7	15	29	53	93	159
−2	1	1	4	9	19	36	65	113	192
−1	1	1	5	11	23	43	77	133	225
0	1	1	6	13	27	50	89	153	258
1	1	1	7	15	31	57	101	173	291
2	1	1	8	17	35	64	113	193	324
3	1	1	9	19	39	71	125	213	357
Differences	0	0	1	2	4	7	12	20	33

Table 5.4: Extended Leonardo sequence with $t = 2$.

j \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
−3	1	1	5	12	26	50	91	159	271
−2	1	1	6	14	30	57	103	179	304
−1	1	1	7	16	34	64	115	199	337
0	1	1	8	18	38	71	127	219	370
1	1	1	9	20	42	78	139	239	403
2	1	1	10	22	46	85	151	259	436
3	1	1	11	24	50	92	163	279	469
Differences	0	0	1	2	4	7	12	20	33

Table 5.5: Extended Leonardo sequence with $t = 3$.

Theorem 5.3.16. *Let $\{a_n\}$ be a sequence of order 2 satisfying the non-homogeneous linear relation:*

$$a_n = a_{n-1} + a_{n-2} + Ck^n,\ \ n \geq 2, \tag{5.26}$$

where $a_0 = 0, a_1 = 1$, $C \neq 0$, $k \neq 0$, and $k^2 - k - 1 \neq 0$. Then

$$a_n = \left(1 - \frac{Ck^3}{k^2-k-1}\right) F_n + \left(1 - \frac{Ck^2}{k^2-k-1}\right) F_{n-1} + \frac{Ck^{n+2}}{k^2-k-1}. \tag{5.27}$$

Proof. The homogeneous solution is

$$a_n = c_1\phi^n + c_2\psi^n,$$

where $\phi = \dfrac{1+\sqrt{5}}{2}$ and $\psi = \dfrac{1-\sqrt{5}}{2}$.

The particular solution can be found using the method of undetermined coefficients. Assume the particular solution is of the form $a_n^* = Ak^n$, where A is a constant. Then

$$\begin{aligned} Ak^n &= Ak^{n-1} + Ak^{n-2} + Ck^n \\ \implies A &= \frac{Ck^2}{k^2-k-1}. \end{aligned}$$

Hence, the general solution to (5.26) is

$$a_n = c_1\phi^n + c_2\psi^n + \frac{Ck^{n+2}}{k^2-k-1}.$$

With $a_0 = a_1 = 1$, we have the following system

$$\begin{cases} 1 &= c_1 + c_2 + \dfrac{Ck^2}{k^2-k-1} \\ 1 &= c_1\phi + c_2\psi + \dfrac{Ck^3}{k^2-k-1} \end{cases} \implies \begin{cases} c_1 + c_2 &= 1 - \dfrac{Ck^2}{k^2-k-1} \\ c_1\phi + c_2\psi &= 1 - \dfrac{Ck^3}{k^2-k-1} \end{cases}.$$

Then

$$c_2(\phi - \psi) = \phi - \frac{Ck^2\phi}{k^2-k-1} - 1 + \frac{Ck^3}{k^2-k-1}$$

$$\implies c_2 = \frac{\phi - 1}{\sqrt{5}} + \frac{Ck^3 - Ck^2\phi}{\sqrt{5}(k^2 - k - 1)}$$
$$= -\frac{\psi}{\sqrt{5}} + \frac{Ck^3 - Ck^2\phi}{\sqrt{5}(k^2 - k - 1)}.$$

Moreover,

$$\begin{aligned}
c_1 &= 1 - \frac{Ck^2}{k^2 - k - 1} + \frac{\psi}{\sqrt{5}} - \frac{Ck^3 - Ck^2\phi}{\sqrt{5}(k^2 - k - 1)} \\
&= \frac{(k^2 - k - 1 - Ck^2)(\phi - \psi) + \psi(k^2 - k - 1) - Ck^3 + Ck^2\phi}{\sqrt{5}(k^2 - k - 1)} \\
&= \frac{-(k+1)(\phi - \psi) + (k^2 - Ck^2)(\phi - \psi) + k^2\psi - (k+1)\psi - Ck^3 + Ck^2\phi}{\sqrt{5}(k^2 - k - 1)} \\
&= \frac{-(k+1)\phi + k^2\left((1 - C)(\phi - \psi) - \alpha k + C\phi + \psi\right)}{\sqrt{5}(k^2 - k - 1)} \\
&= \frac{-(k+1)\phi + k^2(\phi + C\psi - Ck)}{\sqrt{5}(k^2 - k - 1)} = \frac{\phi}{\sqrt{5}} - \frac{Ck^3 - Ck^2\psi}{\sqrt{5}(k^2 - k - 1)}.
\end{aligned}$$

Hence, the general solution to (5.26) is

$$\begin{aligned}
a_n &= \left(\frac{\phi}{\sqrt{5}} - \frac{Ck^3 - Ck^2\psi}{\sqrt{5}(k^2 - k - 1)}\right)\phi^n - \left(\frac{\psi}{\sqrt{5}} - \frac{Ck^3 - Ck^2\phi}{\sqrt{5}(k^2 - k - 1)}\right)\psi^n + \frac{Ck^{n+2}}{k^2 - k - 1} \\
&= \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}} - \frac{Ck^3(\phi^n - \psi^n)}{\sqrt{5}(k^2 - k - 1)} + \frac{Ck^2\phi\psi(\phi^{n-1} - \psi^{n-1})}{\sqrt{5}(k^2 - k - 1)} + \frac{Ck^{n+2}}{k^2 - k - 1} \\
&= F_{n+1} - \frac{Ck^3}{k^2 - k - 1}F_n - \frac{Ck^2}{k^2 - k - 1}F_{n-1} + \frac{Ck^{n+2}}{k^2 - k - 1} \\
&= \left(1 - \frac{Ck^3}{k^2 - k - 1}\right)F_n + \left(1 - \frac{Ck^2}{k^2 - k - 1}\right)F_{n-1} + \frac{Ck^{n+2}}{k^2 - k - 1}. \square
\end{aligned}$$

Corollary 5.3.17. *(Shannon and Deveci [74]). Consider a sequence* $\{a_{j,n}\}$ *of order* 2 *satisfying the following non-homogeneous linear recurrence relation:*

$$a_{j,n} = a_{j,n-1} + a_{j,n-2} + (-1)^n j, \; n \geq 2, \; j \geq 0, \tag{5.28}$$

where $a_0 = 0, \; = a_1 = 1$. *Then*

$$a_{j,n} = F_{n+1} + jF_{n-2} + (-1)^n j, \; n \geq 2. \tag{5.29}$$

Proof. Let $C = j$ and $k = -1$. Then

$$\begin{aligned} a_n &= \left(1 - \frac{-j}{1}\right) F_n + \left(1 - \frac{j}{1}\right) F_{n-1} + \frac{(-1)^{n+2} j}{1} \\ &= (1+j)F_n + (1-j)F_{n-1} + (-1)^n j \\ &= F_{n+1} + jF_{n-2} + (-1)^n j. \square \end{aligned}$$

Corollary 5.3.18. *(Shannon and Deveci [74]). Consider a sequence $\{a_n\}$ of order 2 satisfying the following non-homogeneous linear recurrence relation:*

$$a_n = a_{n-1} + a_{n-2} + (-1)^n, \ n \geq 2, \tag{5.30}$$

where $a_0 = 0, a_1 = 1$. Then

$$a_n = 2F_n + (-1)^n. \tag{5.31}$$

Proof. Let $j = 1$ in the previous corollary. Then

$$a_n = F_{n+1} + F_{n-2} + (-1)^n = F_n + F_{n-1} + F_n - F_{n-1} + (-1)^n = 2F_n + (-1)^n. \qquad \square$$

Corollary 5.3.19. *(Shannon and Deveci [74]). Consider a sequence $\{a_{j,n}\}$ of order 2 satisfying the following non-homogeneous linear recurrence relation: Let*

$$a_{j,n} = a_{j,n-1} + a_{j,n-2} + (-1)^n j, \ n \geq 2, \ j \geq 0,$$

where $a_0 = 0, \ = a_1 = 1$. Then

$$a_{j+1,n} - a_{j,n} = F_{n-2} + (-1)^n, \ n \geq 2. \tag{5.32}$$

Proof. By Corollary 5.3.17, we have

$$a_{j,n} = F_{n+1} + jF_{n-2} + (-1)^n j$$

and

$$a_{j+1,n} = F_{n+1} + (j+1)F_{n-2} + (-1)^n(j+1).$$

Then

$$a_{j+1,n} - a_{j,n} = F_{n-2} + (-1)^n. \qquad \square$$

j \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	1	1	2	3	5	8	13	21	34
1	0	1	2	2	5	6	12	17	30	46
2	0	1	3	2	7	7	16	21	39	58
3	0	1	4	2	9	8	20	25	48	70
4	0	1	5	2	11	9	24	29	57	82
5	0	1	6	2	13	10	28	33	66	94
6	0	1	7	2	15	11	32	37	75	106
7	0	1	8	2	17	12	36	41	84	118
8	0	1	9	2	19	13	40	45	93	130
9	0	1	10	2	21	14	44	49	102	142
10	0	1	11	2	23	15	48	53	111	154
Differences	0	0	1	0	2	1	4	4	9	12

Table 5.6: Table of values for Corollary 3.14.

5.3.2 Examples

Consider

$$w_n(w_0, w_1, p, q, 0, 0) = pw_{n-1}(w_0, w_1, p, q, 0, 0) + qw_{n-2}(w_0, w_1, p, q, 0, 0), \;\; n \geq 2,$$

where w_0, w_1, p, and $q \neq 0$ are given constants. In this section, examples using some common sequences listed in Table 5.1 in Section 5.2 are presented.

Example 5.3.1. *Let* $w_0 = 2,\ w_1 = 1,\ p = q = 1$ *in* (5.13), *i.e.,*

$$w_n(2,1,1,1,t,j) = w_{n-1}(2,1,1,1,t,j) + w_{n-2}(2,1,1,1,t,j) + tn + j,\ n \geq 2,\ t \in \mathbb{Z},$$

Then

(1) $w_n(2,1,1,1,t,j) = L_n + (j+3t)\left(F_{n+1} - 1\right) + t\left(F_n - n\right)$*;*

(2) $w_n(2,1,1,1,t,j+1) - w_n(2,1,1,1,t,j) = F_{n+1} - 1.$

Proof. Since $p = q = 1,\ \alpha = \dfrac{1+\sqrt{5}}{2},\ \beta = \dfrac{1-\sqrt{5}}{2},\ w_n(w_0, w_1, p, q, 0, 0) = w_n(2,1,1,1,0,0) = L_n,\ w_n(0,1,p,q,0,0) = w_n(0,1,1,1,0,0) = F_n,$ and $w_n(1,1,p,q,0,0) = w_n(1,1,1,1,0,0) = F_{n+1}.$ Then by Theorem 5.3.7,

$$\begin{aligned} w_n(2,1,1,1,t,j) &= w_n(2,1,1,1,0,0) + \left(j - \frac{t(1+2)}{1-1-1}\right)\left(w_n(1,1,1,1,0,0) - 1\right) \\ &\quad + t\left(w_n(0,1,1,1,0,0) - n\right) \\ &= L_n + (j+3t)\left(F_{n+1} - 1\right) + t\left(F_n - n\right). \end{aligned}$$

We use Theorem 5.3.13 to obtain the second result. We have

$$\begin{aligned} w_n(2,1,1,1,t,j+1) - w_n(2,1,1,1,t,j) &= w_n(1,1,1,1,0,0) - 1 \\ &= F_{n+1} - 1. \square \end{aligned}$$

We give three tables to show this difference. For $t = 1$:

j \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
−3	2	1	2	3	6	11	20	35	60
−2	2	1	3	5	10	18	32	55	93
−1	2	1	4	7	14	25	44	75	126
0	2	1	5	9	18	32	56	95	159
1	2	1	6	11	22	39	68	115	192
2	2	1	7	13	26	46	80	135	225
3	2	1	8	15	30	53	92	155	258
Differences	0	0	1	2	4	7	12	20	33

Table 5.7: Values of $w_n(2,1,1,1,1,j)$.

For $t = 2$:

j \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
−3	2	1	4	8	17	32	58	101	172
−2	2	1	5	10	21	39	70	121	205
−1	2	1	6	12	25	46	82	141	238
0	2	1	7	14	29	53	94	161	271
1	2	1	8	16	33	60	106	181	304
2	2	1	9	18	37	67	118	201	337
3	2	1	10	20	41	74	130	221	370
Differences	0	0	1	2	4	7	12	20	33

Table 5.8: Values of $w_n(2,1,1,1,2,j)$.

For $t = 3$:

n \ j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
-3	2	1	6	13	28	53	96	167	284
-2	2	1	7	15	32	60	108	187	317
-1	2	1	8	17	36	67	120	207	350
0	2	1	9	19	40	74	132	227	383
1	2	1	10	21	44	81	144	247	416
2	2	1	11	23	48	88	156	267	449
3	2	1	12	25	52	95	168	287	482
Differences	0	0	1	2	4	7	12	20	33

Table 5.9: Values of $w_n(2,1,1,1,3,j)$.

We can see that the difference resembles the sequence $\{F_{n+1} - 1\}$.

Example 5.3.2. *Let $w_0 = 0$, $w_1 = 1$, $p = 2$, and $q = 1$ in (5.13), i.e.,*

$$w_n(0,1,2,1,t,j) = 2w_{n-1}(0,1,2,1,t,j) + w_{n-2}(0,1,2,1,t,j) + 2(tn + j),\ n \geq 2,\ t \in \mathbb{Z}.$$

Then

(1). $w_n(0,1,2,1,t,j) = (1+t)P_n + (j+2t)(P_{n+1} - P_n - 1) - tn$;

(2). $w_n(0,1,2,1,t,j+1) - w_n(0,1,2,1,t,j) = P_{n+1} - P_n - 1$.

Proof. Since $p = 2$ and $q = 1$, $\alpha = 1 + \sqrt{2}$, $\beta = 1 - \sqrt{2}$, $w_n(w_0, w_1, p, q, 0, 0) = w_n(0,1,2,1,0,0) = P_n$. Then by Theorem 5.3.7,

$$w_n(0,1,2,1,t,j) = w_n(0,1,2,1,0,0) + \left(j - \frac{t(2+2)}{1-2-1}\right)(w_n(1,1,2,1,0,0) - 1)$$

$$+\,t\,(w_n(0,1,2,1,0,0)-n)$$

$$= P_n + (j+2t)\,(P_{n+1} - P_n - 1) + tP_n - tn$$

$$= (1+t)P_n + (j+2t)(P_{n+1} - P_n - 1) - tn.$$

We use Theorem 5.3.13 to obtain the second result. Then

$$w_n(0,1,2,1,t,j+1) - w_n(0,1,2,1,t,j) = w_n(1,1,2,1,0,0) - 1 = P_{n+1} - P_n - 1. \qquad \square$$

Remark 5.3.3. *In Example 5.3.2, the following identity is used:*

$$w_n(1,1,2,1,0,0) = w_{n+1}(0,1,2,1,0,0) - w_n(0,1,2,1,0,0).$$

Proof.

$$\begin{aligned}
w_n(0,1,2,1,0,0) &= \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \\
w_{n+1}(0,1,2,1,0,0) &= \frac{(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}}{2\sqrt{2}} \\
w_{n+1}(0,1,2,1,0,0) - w_n(0,1,2,1,0,0) &= \frac{(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}}{2\sqrt{2}} \\
&\quad - \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \\
&= \frac{\sqrt{2}(1+\sqrt{2})^n + \sqrt{2}(1-\sqrt{2})^n}{2\sqrt{2}} \\
&= \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2} \\
&= w_n(1,1,2,1,0,0). \square
\end{aligned}$$

Example 5.3.4. *Let $w_0 = 2$, $w_1 = 2$, $p = 2$, and $q = 1$ in (5.13), i.e.,*

$$w_n(2,2,2,1,t,j) = 2w_{n-1}(2,2,2,1,t,j) + w_{n-2}(2,2,2,1,t,j) + 2(tn+j), \;\; n \geq 2, \;\; t \in \mathbb{Z}.$$

Then

(1). $w_n(2,2,2,1,t,j) = Q_n + (j+2t)(P_{n+1} - P_n - 1) + t(P_n - n)$;

(2). $w_n(2,2,2,1,t,j+1) - w_n(2,2,2,1,t,j) = P_{n+1} - P_n - 1$.

Proof. Similar to the last example, $\alpha = 1+\sqrt{2}$, $\beta = 1-\sqrt{2}$, $w_n(w_0, w_1, p, q, 0, 0) = w_n(2,2,2,1,0,0) = Q_n$ and $w_n(0,1,2,1,0,0) = P_n$. Then by Theorem 5.3.7,

$$\begin{aligned} w_n(2,2,2,1,t,j) &= w_n(2,2,2,1,0,0) + \left(j - \frac{t(2+2)}{1-2-1}\right)(w_n(1,1,2,1,0,0) - 1) \\ &\quad + t\,(w_n(0,1,2,1,0,0) - n) \\ &= Q_n + (j+2t)\,(P_{n+1} - P_n - 1) + t(P_n - n). \end{aligned}$$

The second result is the same as the last example. □

Example 5.3.5. *Let* $w_0 = 0$, $w_1 = 1$, $p = 1$, *and* $q = 2$ *in* (5.13), *i.e.,*

$$w_n(0,1,1,2,t,j) = w_{n-1}(0,1,1,2,t,j) + 2w_{n-2}(0,1,1,2,t,j) + 2(tn+j), \;\; n \geq 2, \;\; t \in \mathbb{Z}.$$

Then

(1). $w_n(0,1,1,2,t,j) = (1+t)J_n + \left(j + \frac{5t}{2}\right)(\mathcal{J}_n - 1) - tn$;

(2). $w_n(0,1,1,2,t,j+1) - w_n(0,1,1,2,t,j) = \mathcal{J}_n - 1$.

Proof. Since $p = 1$ and $q = 2$, we have $\alpha = -1$, $\beta = 2$, and $w_n(w_0, w_1, p, q, 0, 0) = w_n(0,1,1,2,0,0) = J_n$. Then by Theorem 5.3.7,

$$\begin{aligned} w_n(0,1,1,2,t,j) &= w_n(0,1,1,2,0,0) + \left(j - \frac{t(1+2\cdot 2)}{1-p-q}\right)(w_n(1,1,1,2,0,0) - 1) \\ &\quad + t\,(w_n(0,1,1,2,0,0) - n) \\ &= J_n + \left(j + \frac{5t}{2}\right)(\mathcal{J}_n - 1) + tJ_n - tn \\ &= (1+t)J_n + \left(j + \frac{5t}{2}\right)(\mathcal{J}_n - 1) - tn. \end{aligned}$$

We use Theorem 5.3.13 to obtain the second result. Then

$$w_n(0,1,1,2,t,j+1) - w_n(0,1,1,2,t,j) = w_n(1,1,2,1,0,0) - 1 = \mathcal{J}_n - 1.\square$$

Remark 5.3.6. *In Example 5.3.5, the following identity is used:*

$$w_{n+1}(0,1,1,2,0,0) = w_n(1,1,1,2,0,0).$$

Proof.

$$\begin{aligned} w_{n+1}(0,1,1,2,0,0) &= \frac{1}{3}\left((-1)^{n+2} + 2^{n+1}\right) = \frac{1}{3}\left((-1)^n + 2^{n+1}\right) \\ &= w_n(1,1,1,2,0,0) = J_{n+1} = \mathcal{J}_n.\square \end{aligned}$$

Example 5.3.7. *Let $w_0 = 1$, $w_1 = 1$, $p = 1$, and $q = 2$ in* (5.13), *i.e.,*

$$w_n(1,1,1,2,t,j) = w_{n-1}(1,1,1,2,t,j) + 2w_{n-2}(1,1,1,2,t,j) + tn + j, \ n \geq 2, \ t \in \mathbb{Z}.$$

Then

(1). $w_n(1,1,1,2,t,j) = \mathcal{J}_n + \left(j + \frac{5t}{2}\right)(\mathcal{J}_n - 1) + t(J_n - n)$*;*

(2). $w_n(1,1,1,2,t,j+1) - w_n(1,1,1,2,t,j) = \mathcal{J}_n - 1$*.*

Proof. Similar to the last example, we have $\alpha = -1$, $\beta = 2$, $w_n(w_0,w_1,p,q,0,0) = w_n(1,1,1,2,0,0) = \mathcal{J}_n$. Then by Theorem 5.3.7,

$$\begin{aligned} w_n(1,1,1,2,t,j) &= w_n(1,1,1,2,0,0) + \left(j - \frac{t(1+2\cdot 2)}{1-1-2}\right)(w_n(1,1,1,2,0,0) - 1) \\ &\quad + t\left(w_n(0,1,1,2,0,0) - n\right) \\ &= \mathcal{J}_n + \left(j + \frac{5t}{2}\right)(\mathcal{J}_n - 1) + t(J_n - n). \end{aligned}$$

The second result is the same as the previous example. $\square$

Remark 5.3.8. *In Examples 5.3.2, 5.3.4, 5.3.5, 5.3.7, the homogeneous parts are Pell sequence*

$\{P_n\}$, *Pell–Lucas sequence* $\{Q_n\}$, *Jacobsthal sequence* $\{J_n\}$, *and Jacobsthal–Lucas sequence* $\{\mathcal{J}_n\}$, *respectively.*

5.3.3 Some identities involving the generalized Leonardo sequence

Theorem 5.3.20. *Let* $\{\mathcal{L}_{k,n}\}$ *denote the generalized Leonardo sequence. Then*

1. *(Shattuck [76]).*

$$\mathcal{L}_{k,n}^2 - \mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} = (-1)^n(k+1)^2 + k(k+1)F_{n-2}$$

2. *(Kuhapatanakul and Chobsorn [45]).*

$$\mathcal{L}_{k,m}\mathcal{L}_{k,n-1} + \mathcal{L}_{k,m-1}\mathcal{L}_{k,n} = \mathcal{L}_{k,m+1}\mathcal{L}_{k,n+1} - (k+1)\mathcal{L}_{k,m+n} - k$$

Proof. By Theorem 5.3.1, we can write the generalized Leonardo sequence as

$$\mathcal{L}_{k,n} = (1+k)F_{n+1} - k. \tag{5.33}$$

Then

$$\begin{aligned}
\mathcal{L}_{k,n}^2 - \mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} &= (1+k)^2F_{n+1}^2 - 2k(1+k)F_{n+1} + k^2 - ((1+k)F_n - k)\left((1+k)F_{n+2} - k\right) \\
&= (1+k)^2\left(F_{n+1}^2 - F_nF_{n+2}\right) - k(k+1)\left(2F_{n+1} - F_n - F_{n+2}\right) \\
&= (1+k)^2(-1)^n - k(1+k)\left(2F_{n+1} - F_n - F_{n+1} - F_n\right) \\
&= (1+k)^2(-1)^n - k(1+k)F_{n-2},
\end{aligned}$$

by Cassini's identity.

For the second result, we first note Honsberger's identity

$$F_{n-1}F_m + F_nF_{m+1} = F_{m+n}.$$

Then

$$\mathcal{L}_{k,m}\mathcal{L}_{k,n-1} = (1+k)^2 F_{m+1}F_n - k(1+k)(F_{m+1}+F_n) + k^2,$$

$$\mathcal{L}_{k,m-1}\mathcal{L}_{k,n} = (1+k)^2 F_m F_{n+1} - k(1+k)(F_m + F_{n+1}) + k^2,$$

$$\begin{aligned}\mathcal{L}_{k,m+1}\mathcal{L}_{k,n+1} &= (1+k)^2 F_{m+2}F_{n+2} - k(1+k)(F_{m+2}+F_{n+2}) + k^2 \\ &= (1+k)^2(F_{m+1}F_{n+1} + F_{m+1}F_n + F_{n+1}F_m + F_m F_n) \\ &\quad - k(1+k)(F_{m+1}+F_m+F_{n+1}+F_n) + k^2, \\ \mathcal{L}_{k,m+n} &= (1+k)F_{m+n+1} - k\end{aligned}$$

Then

$$\begin{aligned}\mathcal{L}_{k,m}\mathcal{L}_{k,n-1}\mathcal{L}_{k,m-1}\mathcal{L}_{k,n} - \mathcal{L}_{k,m+1}\mathcal{L}_{k,n+1} &= k^2 - (1+k)^2(F_{m+1}F_{n+1} + F_m F_n) \\ &= k^2 - (1+k)^2 F_{m+n+1}.\end{aligned}$$

Finally,

$$\mathcal{L}_{k,m}\mathcal{L}_{k,n-1}\mathcal{L}_{k,m-1}\mathcal{L}_{k,n} - \mathcal{L}_{k,m+1}\mathcal{L}_{k,n+1} + (1+k)\mathcal{L}_{k,m+n} + k = 0. \qquad \square$$

Theorem 5.3.21. *Let*

$$a_0 F_{n+t} + a_1 F_{n+t-1} + \cdots + a_t F_n = 0, \tag{5.34}$$

where $a_0 + a_1 + \cdots + a_t = 0$, $a_i \in \mathbb{Z}$, $(i = 0, 1, 2, \ldots, t)$, t *is a fixed positive integer. Then*

$$a_0 \mathcal{L}_{k,n+t-1} + a_1 \mathcal{L}_{k,n+t-2} + \cdots + a_t \mathcal{L}_{k,n-1} = 0. \tag{5.35}$$

Proof. Since $\mathcal{L}_{k,n} = (1+k)F_{n+1} - k$, we have

$$\begin{aligned}&a_0 \mathcal{L}_{k,n+t-1} + a_1 \mathcal{L}_{k,n+t-2} + \cdots + a_t \mathcal{L}_{k,n-1} \\ &\quad = a_0[(1+k)F_{n+t} - k] + a_1[(1+k)F_{n+t-1} - k] + \cdots + a_t[(1+k)F_n - k] \\ &\quad = (1+k)[a_0 F_{n+t} + a_1 F_{n+t-1} + \cdots + a_t F_n] - k[a_0 + a_1 + \cdots + a_t]\end{aligned}$$

$= (1 + k) \cdot 0 - k \cdot 0 = 0.\square$

Remark 5.3.9. (5.34) *can be obtained by computing* $(x^2 - x - 1)x^n(x - 1)p(x)$, *where* $p(x)$ *is a polynomial over* $\mathbb{Z}$ *first, then replace each* x^{n+i} *by* F_{n+i}.

Algorithm 16 Obtaining this identity

Input: A polynomial $p(x)$ over $\mathbb{Z}$

Output: An identity with generalized Leonardo sequence

1: $g(x) \leftarrow (x^2 - x - 1) \cdot x^n \cdot (x - 1) \cdot p(x)$

2: Replace each x^{n+i} by F_{n+i}

3: Verify the coefficients of F_{n+i} sums to zero

4: Replace each F_{n+i} by $\mathcal{L}_{n+i-1}$

5: Ouput the identity

Example 5.3.10. *It is known that*

$$F_n + F_{n+1} + F_{n+6} - 3F_{n+4} = 0.$$

Hence $a_0 = 1$, $a_1 = 0$, $a_2 = -3$, $a_3 = a_4 = 0$, $a_5 = 1$, $a_6 = 1$, *i.e.* $\sum a_i = 0$. *Then*

$$\mathcal{L}_{k,n+5} - 3\mathcal{L}_{k,n+3} + \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} = 0,$$

or

$$\mathcal{L}_{k,n+5} + \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} = 3\mathcal{L}_{k,n+3}. \tag{5.36}$$

Example 5.3.11. *Let* $f(x) = (x^2 - x - 1)x^n$ *and* $p(x) = (x - 1)(2x^3 + 3x - 1)$. *Then* $g(x) = f(x) \cdot p(x) = 2x^{n+6} - 4x^{n+5} + 3x^{n+4} - 5x^{n+3} + 2x^{n+2} + 3x^{n+1} - x^n$. *Replacing each* x^{n+i} *by* F_{n+i}, *we have*

$$2F_{n+6} - 4F_{n+5} + 3F_{n+4} - 5F_{n+3} + 2F_{n+2} + 3F_{n+1} - F_n = 0. \qquad (5.37)$$

The coefficients are

$$a_0 = 2,\ a_1 = -4,\ a_2 = 3,\ a_3 = -5,\ a_4 = 2,\ a_5 = 3,\ a_6 = -1,$$

which gives

$$\sum a_i = 0.$$

Then we have

$$2\mathcal{L}_{k,n+5} - 4\mathcal{L}_{k,n+4} + 3\mathcal{L}_{k,n+3} - 5\mathcal{L}_{k,n+2} + 2\mathcal{L}_{k,n+1} + 3\mathcal{L}_{k,n} - \mathcal{L}_{k,n-1} = 0,\ n \geq 1.$$

Example 5.3.12. *Let* $f(x) = (x^2 - x - 1)x^n$ *and let* $p(x) = (x-1)(2x^2 + x + 1)$*. Then* $g(x) = f(x) \cdot p(x) = 2x^{n+5} - 3x^{n+4} - x^{n+3} + x^{n+1} + x^n$*. Replacing each* x^{n+i} *by* F_{n+i}*, we have*

$$2F_{n+5} - 3F_{n+4} - F_{n+3} + F_{n+1} + F_n = 0. \qquad (5.38)$$

The coefficients are

$$a_0 = 2,\ a_1 = -3,\ a_2 = -1,\ a_3 = 0,\ a_4 = 1,\ a_5 = 1,$$

which gives

$$\sum a_i = 0.$$

Then we have

$$2\mathcal{L}_{k,n+4} - 3\mathcal{L}_{k,n+3} - \mathcal{L}_{k,n+2} + \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} = 0,\ n \geq 1.$$

5.4 Series $\sum_{n=0}^{\infty} \frac{a_{mn}}{10^{n+1}}$

5.4.1 Main Results

In our main theorem, we will use the following lemma:

Lemma 5.4.1. *Suppose* $|t| < 1$. *Then*

$$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}.$$

Theorem 5.4.1. *(Liu and Shiue [54]). Let* $\{a_n\}$ *satisfy the second order linear recurrence sequence defined in* (5.2). *Suppose* α *and* β *are two different roots to the characteristic equation* $x^2 - px - q = 0$, k *is a positive integer, and* $r > 1$ *satisfying*

$$k < \left\lceil \frac{\log r}{\log(\max\{|\alpha|, |\beta|\})} \right\rceil. \tag{5.39}$$

Then

$$\sum_{n=0}^{\infty} \frac{a_{nk}}{r^{n+1}} = \frac{a_0 r - a_0 \cdot \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} + a_1 \cdot \frac{\alpha^k - \beta^k}{\alpha - \beta}}{(r - \alpha^k)(r - \beta^k)}. \tag{5.40}$$

Proof. From Theorem 5.3.4 and Lemma 5.4.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{a_{nk}}{r^{n+1}} &= \frac{1}{r} \sum_{n=0}^{\infty} \left[\left(\frac{a_1 - \beta a_0}{\alpha - \beta} \right) \frac{\alpha^{nk}}{r^n} - \left(\frac{a_1 - \alpha a_0}{\alpha - \beta} \right) \frac{\beta^{nk}}{r^n} \right] \\
&= \frac{1}{r} \left[\left(\frac{a_1 - \beta a_0}{\alpha - \beta} \right) \sum_{n=0}^{\infty} \frac{\alpha^{nk}}{r^n} - \left(\frac{a_1 - \alpha a_0}{\alpha - \beta} \right) \sum_{n=0}^{\infty} \frac{\beta^{nk}}{r^n} \right] \\
&= \frac{1}{r} \left[\left(\frac{a_1 - \beta a_0}{\alpha - \beta} \right) \frac{r}{r - \alpha^k} - \left(\frac{a_1 - \alpha a_0}{\alpha - \beta} \right) \frac{r}{r - \beta^k} \right] \\
&= \frac{1}{\alpha - \beta} \left[\frac{(\alpha - \beta) a_0 r - a_0(\alpha^{k+1} - \beta^{k+1}) + a_1(\alpha^k - \beta^k)}{(r - \alpha^k)(r - \beta^k)} \right] \\
&= \frac{a_0 r - a_0 \cdot \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} + a_1 \cdot \frac{\alpha^k - \beta^k}{\alpha - \beta}}{(r - \alpha^k)(r - \beta^k)}
\end{aligned}$$

Note that Lemma 3.1 can be used because (5.39) is equivalent to

$$\max\left\{\frac{|\alpha|^k}{r}, \frac{|\beta|^k}{r}\right\} < 1,$$

which is necessary criterion for the convergence of the present geometric series. □

If the characteristic equation $x^2 - px - q = 0$ has repeated roots, we have a similar result.

Lemma 5.4.2. *Suppose $|t| < 1$. Then*

$$\sum_{n=1}^{\infty} nt^n = \frac{t}{(1-t)^2}.$$

Proof. Differentiate the geometric series in Lemma 5.4.1 with respect to t, we have

$$\sum_{n=1}^{\infty} nt^{n-1} = \frac{1}{(1-t)^2}.$$

Hence,

$$\sum_{n=1}^{\infty} nt^n = \frac{t}{(1-t)^2}.$$

□

Theorem 5.4.2. *Let $\{a_n\}$ be satisfied the second order linear recurrence sequence defined in (5.2). Suppose α is a repeated root to the characteristic equation $x^2-px-q=0$ and satisfy $k < \left\lceil \frac{\log r}{\log|\alpha|} \right\rceil$. Then*

$$\sum_{n=0}^{\infty} \frac{a_{nk}}{r^{n+1}} = \frac{a_0 r - (k+1)a_0\alpha^k + ka_1\alpha^{k-1}}{(r-\alpha^k)^2}, \tag{5.41}$$

where k is a positive integer and $r > 1$.

Proof. By Theorem 5.3.4 and Corollary 5.4.2, we have

$$\sum_{n=0}^{\infty} \frac{a_{nk}}{r^{n+1}} = \sum_{n=0}^{\infty} \frac{nka_1\alpha^{nk-1}}{r^{n+1}} - \sum_{n=0}^{\infty} \frac{(nk-1)a_0\alpha^{nk}}{r^{n+1}}$$

$$
\begin{aligned}
&= \frac{ka_1}{\alpha r}\sum_{n=1}^{\infty} n\left(\frac{\alpha^k}{r}\right)^n - \frac{ka_0}{r}\sum_{n=1}^{\infty} n\left(\frac{\alpha^k}{r}\right)^n + \frac{a_0}{r}\sum_{n=0}^{\infty}\left(\frac{\alpha^k}{r}\right)^n \\
&= \frac{ka_1}{\alpha r}\cdot\frac{\frac{\alpha^k}{r}}{\left(1-\frac{\alpha^k}{r}\right)^2} - \frac{ka_0}{r}\cdot\frac{\frac{\alpha^k}{r}}{\left(1-\frac{\alpha^k}{r}\right)^2} + \frac{a_0}{r}\cdot\frac{1}{1-\frac{\alpha^k}{r}}r \\
&= \frac{ka_1\alpha^{k-1}}{(r-\alpha^k)^2} - \frac{ka_0\alpha^k}{(r-\alpha^k)^2} + \frac{a_0}{r-\alpha^k} \\
&= \frac{a_0 r - (k+1)a_0\alpha^k + ka_1\alpha^{k-1}}{(r-\alpha^k)^2}
\end{aligned}
$$

Note that the condition $k < \left\lceil \frac{\log r}{\log|\alpha|} \right\rceil$ if and only if $\frac{|\alpha|^k}{r} < 1$, which guarantees convergence of the geometric series seen in the above steps. We then impose the ceiling function since k is a positive integer. □

Remark 5.4.1. *In future proofs, the equivalence between* $k < \left\lceil \frac{\log r}{\log|\alpha|} \right\rceil$ *and* $\frac{|\alpha|^k}{r} < 1$ *will be exercised implicitly, trusting that the reader will recognize when it is used. Similarly, the equivalence between* $k < \left\lceil \frac{\log r}{\log(\max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|\})} \right\rceil$ *and* $\max\left\{\frac{|\alpha_1|^k}{r}, \frac{|\alpha_2|^k}{r}, \ldots, \frac{|\alpha_n|^k}{r}\right\} < 1$ *will be used in the later sections.*

Remark 5.4.2. *Let* α *be fixed and* $\beta \to \alpha$ *in the right hand side of* (5.40)*. Since*

$$
\lim_{\beta\to\alpha} \frac{\alpha^k - \beta^k}{\alpha - \beta} = \lim_{\beta\to\alpha} \sum_{j=0}^{k-1} \alpha^j \beta^{k-j-1} = k\alpha^{k-1},
$$

then the limit of the right hand side of (5.40) *is the same as* (5.41)*.*

Example 5.4.1. *Let* $a_n = n$*, where* $a_0 = 0$*,* $a_1 = 1$*, and* a_n *satisfies*

$$
a_n = 2a_{n-1} - a_{n-2}, \ n \geq 2.
$$

The characteristic equation is $x^2 - 2x + 1 = 0$ *has two repeated roots. By Theorem 5.4.2, we have*

$$
\sum_{n=1}^{\infty} \frac{nk}{r^{n+1}} = \frac{k}{(r-1)^2}, \ r > 1.
$$

For example, let $k = 1$ *and* $r = 2$*, we obtain* $\sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = 1$ *that leads to* $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2.$

Example 5.4.2. *Let $\{a_n\}$ be an arithmetic sequence with $a_0 = 0$ and common difference d such that $a_n = a + nd$. Since $\{a_n\}$ satisfies*

$$a_n = 2a_{n-1} - a_{n-2},\ n \geq 2,$$

we have by Theorem 5.4.2 that,

$$\sum_{n=0}^{\infty} \frac{a+nkd}{r^{n+1}} = \frac{a(r-1)+kd}{(r-1)^2},\ r > 1.$$

If we let $r = 2$, we obtain

$$\sum_{n=1}^{\infty} \frac{a+nkd}{2^{n+1}} = a + kd.$$

When $k = d = 1$ and $a = 0$, we have the result in Example 3.1.

Remark 5.4.3. *Example 5.4.2 can be used to find a closed expression for*

$$\sum_{n=1}^{\infty} \frac{n^2}{r^{n+1}}.$$

We start with the calculation

$$\begin{aligned}(r-1)\sum_{n=1}^{\infty} \frac{n^2}{r^{n+1}} &= \sum_{n=1}^{\infty} \left(\frac{n^2}{r^n} - \frac{n^2}{r^{n+1}}\right) \\ &= \frac{1}{r} + \sum_{n=1}^{\infty} \frac{(n+1)^2 - n^2}{r^{n+1}} \\ &= \frac{1}{r} + \sum_{n=1}^{\infty} \frac{2n+1}{r^{n+1}} \\ &= \frac{1}{r} + \frac{r+1}{(r-1)^2}\end{aligned}$$

where the last step comes from applying Example 5.4.2 with $a = k = 1$ and $d = 2$. Then

$$\sum_{n=1}^{\infty} \frac{n^2}{r^{n+1}} = \frac{1}{r-1}\left(\frac{1}{r} + \frac{r+1}{(r-1)^2}\right) = \frac{2r^2 - r + 1}{r(r-1)^3}.$$

Remark 5.4.4. *Example 5.4.2 and Remark 5.4.3 can also be used to find a closed expression for*

$$\sum_{n=1}^{\infty} \frac{n^3}{r^{n+1}}.$$

A similar process to Remark 5.4.2 yields

$$\begin{aligned}
(r-1)\sum_{n=1}^{\infty} \frac{n^3}{r^{n+1}} &= \sum_{n=1}^{\infty}\left(\frac{n^3}{r^n} - \frac{n^3}{r^{n+1}}\right) \\
&= \frac{1}{r} + \sum_{n=1}^{\infty} \frac{(n+1)^3 - n^3}{r^{n+1}} \\
&= \frac{1}{r} + \sum_{n=1}^{\infty} \frac{3n^2+3n+1}{r^{n+1}} \\
&= \frac{1}{r} + 3\sum_{n=1}^{\infty} \frac{n^2}{r^{n+1}} + \sum_{n=1}^{\infty} \frac{3n+1}{r^{n+1}} \\
&= \frac{1}{r} + 3\cdot\frac{2r^2-r+1}{r(r-1)^3} + \frac{r+2}{(r-1)^3},
\end{aligned}$$

where the last step comes from applying Remark 5.4.2 and Example 5.4.2 with $a = k = 1$ and $d = 3$. Then

$$\sum_{n=1}^{\infty} \frac{n^3}{r^{n+1}} = \frac{2r^3+4r^2-2r+4}{r(r-1)^4}.$$

Remarks 5.4.3 and 5.4.4 culminate into the following theorem that generalizes inductively to integral powers.

Theorem 5.4.3. *Let k, r with k a positive integer and $r > 1$. Then*

$$(r-1)^2\sum_{n=1}^{\infty} \frac{n^k}{r^n} = (r-1)\left(k\sum_{n=1}^{\infty}\frac{n^{k-1}}{r^n} + \binom{k}{2}\sum_{n=1}^{\infty}\frac{n^{k-2}}{r^n} + \cdots + k\sum_{n=1}^{\infty}\frac{n}{r^n}\right) + r.$$

In particular, if $r = 2$, then

$$\sum_{n=1}^{\infty}\frac{n^k}{2^n} = k\sum_{n=1}^{\infty}\frac{n^{k-1}}{2^n} + \binom{k}{2}\sum_{n=1}^{\infty}\frac{n^{k-2}}{2^n} + \cdots + k\sum_{n=1}^{\infty}\frac{n}{2^n} + 2$$

Proof. Let S be the series,

$$S = \sum_{n=1}^{\infty}\frac{n^k}{r^n} = \frac{1^k}{r} + \frac{2^k}{r^2} + \cdots.$$

Treating S as a power series $\sum n^k z^n$ where $z = \frac{1}{r}$, one can easily use the ratio test to find that the radius of convergence is 1. Thus, S converges for $|z| < 1$, or $r > 1$.

Then algebraic manipulation begins by noting that

$$rS - S = 1^k + \frac{2^k - 1^k}{r} + \frac{3^k - 2^k}{r^2} + \cdots$$

and so,

$$\begin{aligned}
(r-1)S &= \sum_{n=1}^{\infty} \frac{(n+1)^k - n^k}{r^n} + 1 \\
&= \sum_{n=1}^{\infty} \frac{\binom{k}{1}n^{k-1} + \binom{k}{2}n^{k-2} + \cdots + \binom{k}{k-1}n + 1}{r^n} + 1 \\
&= k\sum_{n=1}^{\infty} \frac{n^{k-1}}{r^n} + \binom{k}{2}\sum_{n=1}^{\infty} \frac{n^{k-2}}{r^n} + \cdots + k\sum_{n=1}^{\infty} \frac{n}{r^n} + \sum_{n=1}^{\infty} \frac{1}{r^n} + 1 \\
&= k\sum_{n=1}^{\infty} \frac{n^{k-1}}{r^n} + \binom{k}{2}\sum_{n=1}^{\infty} \frac{n^{k-2}}{r^n} + \cdots + k\sum_{n=1}^{\infty} \frac{n}{r^n} + \frac{1}{r-1} + 1.
\end{aligned}$$

Hence,

$$(r-1)^2 S = (r-1)^2 \sum_{n=1}^{\infty} \frac{n^k}{r^n} = (r-1)\left(k\sum_{n=1}^{\infty} \frac{n^{k-1}}{r^n} + \binom{k}{2}\sum_{n=1}^{\infty} \frac{n^{k-2}}{r^n} + \cdots + k\sum_{n=1}^{\infty} \frac{n}{r^n}\right) + r.$$

□

Example 5.4.3. *Making the following substitutions into Theorem 5.4.3, we calculate*

$$\begin{aligned}
\mathbf{r = 2, k = 2}: &\quad \sum_{n=1}^{\infty} \frac{n^2}{2^n} = 2\sum_{n=1}^{\infty} \frac{n}{r^n} + 2 = 6 \\
\mathbf{r = 2, k = 3}: &\quad \sum_{n=1}^{\infty} \frac{n^3}{2^n} = 3\sum_{n=1}^{\infty} \frac{n^2}{2^n} + 3\sum_{n=1}^{\infty} \frac{n}{r^n} + 2 = 26 \\
\mathbf{r = 2, k = 4}: &\quad \sum_{n=1}^{\infty} \frac{n^4}{2^n} = 4\sum_{n=1}^{\infty} \frac{n^3}{2^n} + 6\sum_{n=1}^{\infty} \frac{n^2}{2^n} + 4\sum_{n=1}^{\infty} \frac{n}{2^n} + 2 = 150.
\end{aligned}$$

Remark 5.4.5. *Let $\{a_n\}$ be a second order linear recurrence sequence satisfying* (5.2). *If $a_1 = \alpha a_0 \neq 0$, where α is a root of the characteristic equation $x^2 - px - q = 0$, then substituting into the* *formula of Theorem 5.3.4, it can be deduced that (regardless of whether the characteristic equation has repeated roots)* $a_n = a_0\alpha^n$*; therefore,* $\{a_n\}$ *forms a geometric sequence. If* k *is a positive integer,* $\left\{\frac{a_{nk}}{r^{n+1}}\right\}$ *also forms a geometric sequence with the common ratio* $\frac{\alpha^k}{r}$*. Hence,* $\sum_{n=0}^{\infty}\frac{a_{nk}}{r^{n+1}}$ *converges if and only if* $\frac{|\alpha|^k}{r} < 1$*.*

On the other hand, if it is known that $\{a_n\}$ *is a geometric sequence satisfying* (5.2) *with the common ratio* d*, substituting* $n = 2$ *in* (5.2) *yields*

$$a_0 d^2 = p(a_0 d) + q a_0.$$

Dividing both sides by a_0*, we obtain* $d^2 = pd + q$*. Hence,* d *is the root of* $x^2 - px - q = 0$*.*

When $k = 1$, the formula $\sum_{n=0}^{\infty}\frac{a_n}{r^{n+1}}$ can be further simplified:

Corollary 5.4.4. *Let* $\{a_n\}$ *satisfy the second order linear recurrence sequence defined in* (5.2)*. Suppose* α *and* β *are two roots to the characteristic equation* $x^2 - px - q = 0$ *such that* $|\alpha|, |\beta| < r$*. Then*

$$\sum_{n=0}^{\infty}\frac{a_n}{r^{n+1}} = \frac{a_0(r-p)+a_1}{r^2 - pr - q}.$$

Proof. In Theorem 5.4.1 and 5.4.2, regardless of whether $x^2 - px - q = 0$ has repeated roots, i.e., $\alpha = \beta$ or $\alpha \neq \beta$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{a_n}{r^{n+1}} &= \frac{a_0 r - a_0(\alpha+\beta) + a_1}{(r-\alpha)(r-\beta)}\\ &= \frac{a_0 r - a_0(\alpha+\beta) + a_1}{r^2 - (\alpha+\beta)r + \alpha\beta}.\end{aligned}$$

Furthermore, from the relationship between the roots and coefficients of the quadratic equation

$$\alpha + \beta = p, \ \alpha\beta = -q,$$

we have

$$\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} = \frac{a_0(r-p)+a_1}{r^2-pr-q}.$$

□

Remark 5.4.6. *Alternatively, we may prove Corollary 5.4.4 in the following way. This way, we do not need to use the roots of the characteristic equation to prove the formula* $\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}$*. However, we will still need the roots to prove the convergence of* $\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}$.

Proof. Let $a_{n+2} = pa_{n+1} + qa_n$. Then

$$\sum_{n=0}^{\infty} \frac{a_{n+2}}{r^{n+1}} = p\sum_{n=0}^{\infty} \frac{a_{n+1}}{r^{n+1}} + q\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}.$$

Since

$$\sum_{n=0}^{\infty} \frac{a_{n+2}}{r^{n+1}} = r^2\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - (ra_0 + a_1),$$
$$\sum_{n=0}^{\infty} \frac{a_{n+1}}{r^{n+1}} = r\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - a_0,$$

we have

$$r^2\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - (ra_0 + a_1) = pr\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - pa_0 + q\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}.$$

Then

$$(r^2 - pr - q)\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} = (r-p)a_0 + a_1.$$

Hence,

$$\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} = \frac{a_0(r-p)+a_1}{r^2-pr-q}.$$

□

The next corollary focuses on the special case $p = q = 1$ and $r = 10$.

Corollary 5.4.5. *Let $\{a_n\}$ satisfy the second order linear recurrence sequence $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Then*

$$\sum_{n=0}^{\infty} \frac{a_n}{10^{n+1}} = \frac{9a_0 + a_1}{89}.$$

Proof. Applying Theorem 5.4.1, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{a_n}{10^{n+1}} &= \frac{10a_0 - a_0 \cdot \frac{\phi^2-\psi^2}{\phi-\psi} + a_1 \cdot \frac{\phi-\psi}{\phi-\psi}}{(10-\phi)(10-\psi)} \\ &= \frac{10a_0 - a_0(\phi+\psi) + a_1}{100 - 10(\phi+\psi) + \phi\psi} \\ &= \frac{9a_0 + a_1}{100 - 10 - 1} \\ &= \frac{9a_0 + a_1}{89}.\end{aligned}$$

□

In particular, if $a_0 = 0$ and $a_1 = 1$, we have $\sum_{n=0}^{\infty} \frac{a_n}{10^{n+1}} = \sum_{n=0}^{\infty} \frac{F_n}{10^{n+1}} = \frac{1}{89}$ (Long [55]).

5.4.2 Lucas Sequence of the First Kind

Definition 5.4.1. *(Niven et al. [63], Sloane et al. [84]). A second-order linear recursive sequence $\{u_n\}$ is called a Lucas sequence of the first kind if it satisfies*

$$u_n = pu_{n-1} + qu_{n-2}, \tag{5.42}$$

for some constants $p, q \neq 0$ and initial conditions $u_0 = 0$, $u_1 = 1$.

Theorem 5.4.6. *Let $\{u_n\}$ be the Lucas sequence of the first kind. Let α and β be two roots of the characteristic equation $x^2 - px - q = 0$ and satisfy*

$$k < \left\lceil \frac{\log r}{\log(\max\{|\alpha|, |\beta|\})} \right\rceil,$$

where $r > 1$. Then

$$\sum_{n=1}^{\infty} \frac{u_{nk}}{r^{n+1}} = \frac{u_k}{r^2 - (\alpha^k + \beta^k)r + (-q)^k}.$$

Proof. Note that

$$k < \left\lceil \frac{\log r}{\log(\max\{|\alpha|, |\beta|\})} \right\rceil \text{ if and only if } \max\left\{\left|\frac{\alpha^k}{r}\right|, \left|\frac{\beta^k}{r}\right|\right\} < 1,$$

which is a necessary criteria for the two theorems used in this proof.

When $\alpha \neq \beta$, we apply Theorem 5.4.1. Then for $u_0 = 0$ and $u_1 = 1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{u_{nk}}{r^{n+1}} &= \frac{\frac{\alpha^k - \beta^k}{\alpha - \beta}}{(r - \alpha^k)(r - \beta^k)} \\
&= \frac{u_k}{(r - \alpha^k)(r - \beta^k)} = \frac{u_k}{r^2 - (\alpha^k + \beta^k)r + (\alpha\beta)^k} \\
&= \frac{u_k}{r^2 - (\alpha^k + \beta^k)r + (-q)^k}.
\end{aligned}$$

On the other hand, if $\alpha = \beta$, we have by Theorem 5.4.2,

$$\sum_{n=1}^{\infty} \frac{u_{nk}}{r^{n+1}} = \frac{k\alpha^{k-1}}{(r - \alpha^k)^2} = \frac{u_k}{r^2 - 2r\alpha^k + \alpha^{2k}}.$$

□

Theorem 5.4.7. *Let $\{u_n\}$ be the Lucas sequence of the first kind. Then*

$$\sum_{n=0}^{\infty} \frac{u_{kn-1}}{r^{n+1}} = \frac{1}{q(\alpha - \beta)} \left[\frac{\alpha(r - \alpha^k) - \beta(r - \beta^k)}{(r - \alpha^k)(r - \beta^k)}\right], k < \left\lceil \frac{\log r}{\log(\max\{|\alpha|, |\beta|\})} \right\rceil,$$

where α, β are roots to the characteristic equation $x^2 - px - q = 0$.

Proof. Let $w_n = u_{n-1}$, then $w_{kn} = u_{kn-1}$ with $w_0 = \dfrac{1}{q}$ and $w_1 = 0$. Then

$$\sum_{n=0}^{\infty} \frac{u_{kn-1}}{r^n} = r \sum_{n=0}^{\infty} \frac{w_{kn}}{r^{n+1}} = r \left[\frac{w_0 r - w_0 \cdot \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} + w_1 \cdot \frac{\alpha^k - \beta^k}{\alpha - \beta}}{(r - \alpha^k)(r - \beta^k)}\right]$$

$$= r\left[\frac{\frac{r}{q}-\frac{1}{q}\cdot\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}}{(r-\alpha^k)(r-\beta^k)}\right]$$

$$= \frac{r}{q(\alpha-\beta)}\left[\frac{r(\alpha-\beta)-(\alpha^{k+1}-\beta^{k+1})}{(r-\alpha^k)(r-\beta^k)}\right]$$

$$= \frac{r}{q(\alpha-\beta)}\left[\frac{\alpha(r-\alpha^k)-\beta(r-\beta^k)}{(r-\alpha^k)(r-\beta^k)}\right]$$

□

Theorem 5.4.8. *Let $\{u_n\}$ be the Lucas sequence of the first kind. Then*

$$\sum_{n=0}^{\infty}\frac{u_{kn+1}}{r^n} = \frac{1}{\alpha-\beta}\left(\frac{\alpha(r-\beta^k)-\beta(r-\alpha^k)}{(r-\alpha^k)(r-\beta^k)}\right), k < \left\lceil\frac{\log r}{\log(\max\{|\alpha|,|\beta|\})}\right\rceil,$$

where α, β are roots to the characteristic equation $x^2 - px - q = 0$.

Proof. Let $w_n = u_{n+1}$ for $n \geq 0$, then $w_{kn} = u_{kn+1}$ for $n, k \geq 0$ with $w_0 = u_1 = 1$ and $w_1 = u_2 = pu_1 + qu_0 = p = \alpha + \beta$. Then using Theorem 5.4.1, we have

$$\sum_{n=0}^{\infty}\frac{u_{kn+1}}{r^{n+1}} = \sum_{n=0}^{\infty}\frac{w_{kn}}{r^{n+1}} = \frac{r-\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}+(\alpha+\beta)\frac{\alpha^k-\beta^k}{\alpha-\beta}}{(r-\alpha^k)(r-\beta^k)}$$

$$= \frac{r(\alpha-\beta)-(\alpha^{k+1}-\beta^{k+1})+(\alpha+\beta)(\alpha^k-\beta^k)}{(\alpha-\beta)(r-\alpha^k)(r-\beta^k)}$$

$$= \frac{1}{\alpha-\beta}\left(\frac{\alpha(r-\beta^k)-\beta(r-\alpha^k)}{(r-\alpha^k)(r-\beta^k)}\right).$$

□

A generalization of Theorem 5.4.8 can be found in Melham and Shannon [58].

Example 5.4.4. *(Long [55]). Let $\{F_n\}$ be the Fibonacci sequence, i.e.,*

$$F_n = F_{n-1} + F_{n-2},$$

with $F_0 = 0$ and $F_1 = 1$. The characteristic equation is $x^2 - x - 1 = 0$. The roots are $\alpha = \dfrac{1+\sqrt{5}}{2}$

and $\beta = \dfrac{1-\sqrt{5}}{2}$. *By Theorem 5.4.6, we have*

$$\sum_{n=0}^{\infty} \frac{F_{nk}}{r^{n+1}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}\left(r - \left(\frac{1+\sqrt{5}}{2}\right)^k\right)\left(r - \left(\frac{1-\sqrt{5}}{2}\right)^k\right)}, \; k < \left\lceil \frac{\log r}{\log\left(\frac{1+\sqrt{5}}{2}\right)} \right\rceil. \tag{5.43}$$

Let $r = 10$, *then only* $k = 1, 2, 3, 4$ *satisfy the condition for convergence. We have*

$$\sum_{n=0}^{\infty} \frac{F_n}{10^{n+1}} = \frac{\sqrt{5}}{\sqrt{5}\left(10 - \frac{1+\sqrt{5}}{2}\right)\left(10 - \frac{1-\sqrt{5}}{2}\right)} = \frac{1}{89}, \tag{5.44}$$

$$\sum_{n=0}^{\infty} \frac{F_{2n}}{10^{n+1}} = \frac{\sqrt{5}}{\sqrt{5}\left(10 - \left(\frac{1+\sqrt{5}}{2}\right)^2\right)\left(10 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right)} = \frac{1}{71}, \tag{5.45}$$

$$\sum_{n=0}^{\infty} \frac{F_{3n}}{10^{n+1}} = \frac{2\sqrt{5}}{\sqrt{5}\left(10 - \left(\frac{1+\sqrt{5}}{2}\right)^3\right)\left(10 - \left(\frac{1-\sqrt{5}}{2}\right)^3\right)} = \frac{2}{59}, \tag{5.46}$$

$$\sum_{n=0}^{\infty} \frac{F_{4n}}{10^{n+1}} = \frac{3\sqrt{5}}{\sqrt{5}\left(10 - \left(\frac{1+\sqrt{5}}{2}\right)^4\right)\left(10 - \left(\frac{1-\sqrt{5}}{2}\right)^4\right)} = \frac{3}{31}. \tag{5.47}$$

Corollary 5.4.9. *Let* $\{F_n\}$ *be the Fibonacci sequence. Then by Theorem 5.4.8,*

$$\sum_{n=0}^{\infty} \frac{F_{kn+1}}{r^{n+1}} = \frac{1}{\sqrt{5}}\left(\frac{\phi(r-\psi^k) - \psi(r-\phi^k)}{(r-\phi^k)(r-\psi^k)}\right), \tag{5.48}$$

where $\phi = \dfrac{1+\sqrt{5}}{2}$, $\psi = \dfrac{1-\sqrt{5}}{2}$, *and* $k < \left\lceil \dfrac{\log r}{\log \phi} \right\rceil$. *Furthermore, if* $r = 10$ *in* (5.48), *then only* $k = 1, 2, 3, 4$ *satisfy the condition for convergence. Thus,*

$$\sum_{n=0}^{\infty} \frac{F_{n+1}}{10^{n+1}} = \frac{1}{\sqrt{5}}\left(\frac{\phi(r-\psi) - \psi(r-\phi)}{(r-\phi)(r-\psi)}\right) = \frac{10}{89}, \tag{5.49}$$

$$\sum_{n=0}^{\infty} \frac{F_{2n+1}}{10^{n+1}} = \frac{1}{\sqrt{5}}\left(\frac{\phi(r-\psi^2) - \psi(r-\phi^2)}{(r-\phi^2)(r-\psi^2)}\right) = \frac{9}{71}, \tag{5.50}$$

$$\sum_{n=0}^{\infty} \frac{F_{3n+1}}{10^{n+1}} = \frac{1}{\sqrt{5}}\left(\frac{\phi(r-\psi^3) - \psi(r-\phi^3)}{(r-\phi^3)(r-\psi^3)}\right) = \frac{9}{59}, \tag{5.51}$$

$$\sum_{n=0}^{\infty} \frac{F_{4n+1}}{10^{n+1}} = \frac{1}{\sqrt{5}}\left(\frac{\phi(r-\psi^4) - \psi(r-\phi^4)}{(r-\phi^4)(r-\psi^4)}\right) = \frac{8}{31}. \tag{5.52}$$

Example 5.4.5. *(Ward [93]). Let* $a_0 = 0$*,* $a_1 = 1$*, and* $a_n = 4a_{n-1} - a_{n-2}$*,* $n \geq 2$*, i.e.,* $\{a_n\} = \{0, 1, 4, 15, 56, \cdots\}$*. The two roots of the characteristic equation* $x^2 - 4x + 1 = 0$ *are* $\alpha = 2 + \sqrt{3}$ *and* $\beta = 2 - \sqrt{3}$*, and satisfy* $\alpha + \beta = 4$ *and* $\alpha\beta = 1$*. Hence, from Theorem 5.4.6, we have*

$$\sum_{n=1}^{\infty} \frac{a_{nk}}{r^{n+1}} = \frac{a_k}{r^2 - (\alpha^k + \beta^k)r + 1}, \tag{5.53}$$

with the condition for convergence $k < \dfrac{\log r}{\log(2 + \sqrt{3})}$*,* $r > 1 \in \mathbb{R}$*. With some calculations, we obtain*

$$\begin{aligned}
\alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta = 14, \\
\alpha^3 + \beta^3 &= (\alpha^2 + \beta^2)(\alpha + \beta) - \alpha\beta(\alpha + \beta) = 52, \\
\alpha^4 + \beta^4 &= (\alpha^3 + \beta^3)(\alpha + \beta) - \alpha\beta(\alpha^2 + \beta^2) = 194.
\end{aligned}$$

In fact, using Girard-Waring formulas (Gould [25], Lidl and Niederreiter [51]), we can directly compute $\alpha^k + \beta^k$ *for larger values of* k*:*

$$\begin{aligned}
\alpha^k + \beta^k &= \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k - m - 1)!k}{(k - 2m)!m!}(-1)^m(\alpha + \beta)^{k-2m}(\alpha\beta)^m \\
&= \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k}{k - m}\binom{k - m}{m}(-1)^m 4^{k-2m}.
\end{aligned}$$

For $k = 1, 2, 3, 4$*, we can express* (5.53) *more clearly:*

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{u_n}{r^{n+1}} &= \frac{1}{r^2 - 4r + 1}, \quad \textit{if } k = 1, \\
\sum_{n=1}^{\infty} \frac{u_{2n}}{r^{n+1}} &= \frac{1}{r^2 - 14r + 1}, \quad \textit{if } k = 2, \\
\sum_{n=1}^{\infty} \frac{u_{3n}}{r^{n+1}} &= \frac{1}{r^2 - 52r + 1}, \quad \textit{if } k = 3, \\
\sum_{n=1}^{\infty} \frac{u_{4n}}{r^{n+1}} &= \frac{1}{r^2 - 194r + 1}, \quad \textit{if } k = 4.
\end{aligned}$$

Example 5.4.6. *Let $\{B_n\}$ be the sequence of balancing numbers satisfying*

$$B_n = 6B_{n-1} - B_{n-2},\ n \geq 2,$$

with $B_0 = 0$ and $B_1 = 1$. The characteristic equation for (5.54) is $x^2 - 6x + 1 = 0$. The roots are $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$. By theorem 5.4.6, we have

$$\sum_{n=1}^{\infty} \frac{B_{nk}}{r^{n+1}} = \frac{B_k}{r^2 - r\left((3+2\sqrt{2})^k + (3-2\sqrt{2})^k\right) + 1},\ k < \left\lceil \frac{\log r}{\log(3+2\sqrt{2})} \right\rceil. \tag{5.54}$$

Let $r = 10$, then only $k = 1$ satisfies the condition for convergence. Thus,

$$\sum_{n=1}^{\infty} \frac{B_n}{10^{n+1}} = \frac{1}{100 - 10(6) + 1} = \frac{1}{41}.$$

Example 5.4.7. *Let $\{P_n\}$ be the sequence of Pell numbers satisfying*

$$P_n = 2P_{n-1} + P_{n-2},\ n \geq 2,$$

with $P_0 = 0$ and $P_1 = 1$. The characteristic equation for (5.55) is $x^2 - 2x - 1 = 0$. The roots are $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. By theorem 5.4.6, we have

$$\sum_{n=1}^{\infty} \frac{P_{nk}}{r^{n+1}} = \frac{P_k}{r^2 - r\left((1+\sqrt{2})^k + (1-\sqrt{2})^k\right) + (-1)^k},\ k < \left\lceil \frac{\log r}{\log(1+\sqrt{2})} \right\rceil. \tag{5.55}$$

Let $r = 10$, then only $k = 1, 2$ satisfy the condition for convergence. Thus,

$$\sum_{n=1}^{\infty} \frac{P_n}{10^{n+1}} = \frac{1}{100 - 10(2) - 1} = \frac{1}{79}, \tag{5.56}$$

$$\sum_{n=1}^{\infty} \frac{P_{2n}}{10^{n+1}} = \frac{2}{100 - 10(6) + 1} = \frac{2}{41}. \tag{5.57}$$

By Theorem 5.4.7, we have

$$\sum_{n=0}^{\infty} \frac{P_{kn-1}}{r^{n+1}} = \frac{1}{2\sqrt{2}} \left[\frac{(1+\sqrt{2})(r - (1+\sqrt{2})^k) - (1-\sqrt{2})(r - (1-\sqrt{2})^k)}{(r - (1+\sqrt{2})^k)(r - (1-\sqrt{2})^k)} \right]. \tag{5.58}$$

Let $r = 10$, then for $k = 2$, we have

$$\sum_{n=0}^{\infty} \frac{P_{2n-1}}{10^{n+1}} = \frac{1}{2\sqrt{2}} \left[\frac{10\sqrt{2}}{41} \right] = \frac{5}{41}. \tag{5.59}$$

By Theorem 5.4.8, we have

$$\sum_{n=0}^{\infty} \frac{P_{kn+1}}{r^{n+1}} = \frac{1}{2\sqrt{2}} \left[\frac{(1+\sqrt{2})(r-(1-\sqrt{2})^k) - (1-\sqrt{2})(r-(1+\sqrt{2})^k)}{(r-(1+\sqrt{2})^k)(r-(1-\sqrt{2})^k)} \right]. \tag{5.60}$$

Let $r = 10$, then for $k = 2$, we have

$$\sum_{n=0}^{\infty} \frac{P_{2n-1}}{10^{n+1}} = \frac{1}{2\sqrt{2}} \left[\frac{18\sqrt{2}}{41} \right] = \frac{9}{41}. \tag{5.61}$$

Remark 5.4.7. *It is known that $B_n = \frac{P_{2n}}{2}$ (Panda and Ray [65]). We have*

$$\sum_{n=1}^{\infty} \frac{\frac{P_{2n}}{2}}{10^{n+1}} = \sum_{n=1}^{\infty} \frac{B_n}{10^{n+1}} = \frac{1}{41}.$$

Example 5.4.8. *Let $\{M_n\}$ be the sequence of Mersenne numbers satisfying*

$$M_n = 3M_{n-1} - 2M_{n-2}, \ n \geq 2,$$

with $M_0 = 0$ and $M_1 = 1$. The characteristic equation is $x^2 - 3x + 2 = 0$. The roots are $\alpha = 2$ and $\beta = 1$. By theorem 5.4.6, we have

$$\sum_{n=1}^{\infty} \frac{M_{nk}}{r^{n+1}} = \frac{M_k}{r^2 - r(2^k+1) + 2^k}, \ k < \left\lceil \frac{\log r}{\log 2} \right\rceil. \tag{5.62}$$

Let $r = 10$, then only $k = 1, 2, 3$ satisfy the condition for convergence. Thus,

$$\sum_{n=1}^{\infty} \frac{M_n}{10^{n+1}} = \frac{1}{100 - 10(3) + 2} = \frac{1}{72}, \tag{5.63}$$

$$\sum_{n=1}^{\infty} \frac{M_{2n}}{10^{n+1}} = \frac{3}{100 - 10(5) + 4} = \frac{3}{54} = \frac{1}{18}, \tag{5.64}$$

$$\sum_{n=1}^{\infty} \frac{M_{3n}}{10^{n+1}} = \frac{7}{100 - 10(9) + 8} = \frac{7}{18}. \tag{5.65}$$

Example 5.4.9. *Let $\{J_n\}$ be the sequence of Jacobsthal numbers satisfying*

$$J_n = J_{n-1} + 2J_{n-2},\ n \geq 2,$$

with $J_0 = 0$ and $J_1 = 1$. The characteristic equation is $x^2 - x - 2 = 0$. The roots are $\alpha = 2$ and $\beta = -1$.

By theorem 5.4.6,

$$\sum_{n=1}^{\infty} \frac{J_{nk}}{r^{n+1}} = \frac{J_k}{r^2 - r\left(2^k + (-1)^k\right) + (-2)^k},\ k < \left\lceil \frac{\log r}{\log 2} \right\rceil. \tag{5.66}$$

Let $r = 10$, then only $k = 1, 2, 3$ satisfy the condition for convergence. Thus,

$$\sum_{n=1}^{\infty} \frac{J_n}{10^{n+1}} = \frac{1}{100 - 10 - 2} = \frac{1}{88}, \tag{5.67}$$

$$\sum_{n=1}^{\infty} \frac{J_{2n}}{10^{n+1}} = \frac{1}{100 - 10(5) + 4} = \frac{1}{54}, \tag{5.68}$$

$$\sum_{n=1}^{\infty} \frac{J_{3n}}{10^{n+1}} = \frac{3}{100 - 10(7) - 8} = \frac{3}{22}. \tag{5.69}$$

5.4.3 Lucas Sequence of the Second Kind

Definition 5.4.2. *(Niven et al. [63]). A second-order linear recursive sequence $\{v_n\}$ is called a Lucas sequence of the second kind if it satisfies*

$$v_n = pv_{n-1} + qv_{n-2}, \tag{5.70}$$

for some constants $p, q \neq 0$ and initial conditions $v_0 = 2$, $v_1 = p$.

Theorem 5.4.10. *Let $\{v_n\}$ be the Lucas sequence of the second kind. Let α and β be two roots of* *the characteristic equation $x^2 - px - q = 0$ and satisfy*

$$k < \left\lceil \frac{\log r}{\log(\max\{|\alpha|, |\beta|\})} \right\rceil.$$

Then

$$\sum_{n=0}^{\infty} \frac{v_{nk}}{r^{n+1}} = \frac{2r - (\alpha^k + \beta^k)}{r^2 - (\alpha^k + \beta^k)r + (-q)^k}. \tag{5.71}$$

Proof. By Theorem 5.4.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{v_{nk}}{r^{n+1}} &= \frac{v_0 r - v_0 \cdot \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} + v_1 \cdot \frac{\alpha^k - \beta^k}{\alpha - \beta}}{(r - \alpha^k)(r - \beta^k)} \\
&= \frac{1}{(\alpha - \beta)(r - \alpha^k)(r - \beta^k)} \Big(2r(\alpha - \beta) - 2(\alpha^{k+1} - \beta^{k+1}) + p(\alpha^k - \beta^k) \Big) \\
&= \frac{1}{(\alpha - \beta)(r - \alpha^k)(r - \beta^k)} \Big(2r(\alpha - \beta) - (\alpha - \beta)(\alpha^k + \beta^k) \Big) \\
&= \frac{2r - (\alpha^k + \beta^k)}{(r - \alpha^k)(r - \beta^k)} = \frac{2r - (\alpha^k + \beta^k)}{r^2 - (\alpha^k + \beta^k)r + (-q)^k}.
\end{aligned}$$

□

Example 5.4.10. *Let $\{L_n\}$ be the sequence of Lucas numbers satisfying*

$$L_n = L_{n-1} + L_{n-2}, \ n \geq 2,$$

with $L_0 = 2$ and $L_1 = 1$. The characteristic equation is $x^2 - x - 1 = 0$. The roots are $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

By Theorem 5.4.10, we have

$$\sum_{n=0}^{\infty} \frac{L_{nk}}{r^{n+1}} = \frac{2r - \left[\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k \right]}{r^2 - \left[\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k \right] r + (-1)^k}, \ k < \left\lceil \frac{\log r}{\log(\frac{1+\sqrt{5}}{2})} \right\rceil. \tag{5.72}$$

Let $r = 10$, then only $k = 1, 2, 3, 4$ satisfy the condition for convergence. Thus,

$$\sum_{n=0}^{\infty} \frac{L_n}{10^{n+1}} = \frac{20-1}{100-10-1} = \frac{19}{89}, \tag{5.73}$$

$$\sum_{n=0}^{\infty} \frac{L_{2n}}{10^{n+1}} = \frac{20-3}{100-30+1} = \frac{17}{71}, \tag{5.74}$$

$$\sum_{n=0}^{\infty} \frac{L_{3n}}{10^{n+1}} = \frac{20-4}{100-40-1} = \frac{16}{59}, \tag{5.75}$$

$$\sum_{n=0}^{\infty} \frac{L_{4n}}{10^{n+1}} = \frac{20-7}{100-70+1} = \frac{13}{31}. \tag{5.76}$$

Example 5.4.11. *Let $\{S_n\}$ be the Fermat sequence of numbers of the second kind satisfying*

$$S_n = 3S_{n-1} - 2S_{n-2},\ n \geq 2,$$

with $S_0 = 2$ and $S_1 = 3$. The characteristic equation is $x^2 - 3x + 2 = 0$. The roots are $\alpha = 2$ and $\beta = 1$.

By Theorem 5.4.10, we have

$$\sum_{n=0}^{\infty} \frac{S_{nk}}{r^{n+1}} = \frac{2r - \left(2^k + 1\right)}{r^2 - \left(2^k + 1\right) r + 2^k},\ k < \left\lceil \frac{\log r}{\log 2} \right\rceil. \tag{5.77}$$

Let $r = 10$, then only $k = 1, 2, 3$ satisfy the condition for convergence. Thus,

$$\sum_{n=0}^{\infty} \frac{S_n}{10^{n+1}} = \frac{20-3}{100-30+2} = \frac{17}{72}, \tag{5.78}$$

$$\sum_{n=0}^{\infty} \frac{S_{2n}}{10^{n+1}} = \frac{20-5}{100-50+4} = \frac{15}{54}, \tag{5.79}$$

$$\sum_{n=0}^{\infty} \frac{S_{3n}}{10^{n+1}} = \frac{20-9}{100-90+8} = \frac{11}{18}. \tag{5.80}$$

Example 5.4.12. *Let $\{Q_n\}$ be the Pell sequence of numbers of the second kind satisfying*

$$Q_n = 2Q_{n-1} + Q_{n-2},\ n \geq 2,$$

with $Q_0 = 2$ and $Q_1 = 2$. The characteristic equation is $x^2 - 2x - 1 = 0$. The roots are $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

By Theorem 5.4.10, we have

$$\sum_{n=0}^{\infty} \frac{Q_{nk}}{r^{n+1}} = \frac{2r - \left((1+\sqrt{2})^k + (1-\sqrt{2})^k\right)}{r^2 - \left((1+\sqrt{2})^k + (1-\sqrt{2})^k\right) r + (-1)^k}, \ k < \left\lceil \frac{\log r}{\log(1+\sqrt{2})} \right\rceil. \tag{5.81}$$

Let $r = 10$, then only $k = 1, 2$ satisfy the condition for convergence. Thus,

$$\sum_{n=0}^{\infty} \frac{Q_n}{10^{n+1}} = \frac{20-2}{100-20-1} = \frac{18}{79}, \tag{5.82}$$

$$\sum_{n=0}^{\infty} \frac{Q_{2n}}{10^{n+1}} = \frac{20-6}{100-60+1} = \frac{14}{41}. \tag{5.83}$$

Example 5.4.13. *Let $\{\mathcal{J}_n\}$ be the Jacobsthal-Lucas sequence of numbers satisfying*

$$\mathcal{J}_n = \mathcal{J}_{n-1} + 2\mathcal{J}_{n-2}, \ n \geq 2,$$

with $\mathcal{J}_0 = 2$ and $\mathcal{J}_1 = 1$. The characteristic equation is $x^2 - x - 2 = 0$. The roots are $\alpha = 2$ and $\beta = -1$.

By Theorem 5.4.10, we have

$$\sum_{n=0}^{\infty} \frac{\mathcal{J}_{nk}}{r^{n+1}} = \frac{2r - \left(2^k + (-1)^k\right)}{r^2 - \left(2^k + (-1)^k\right) r + (-2)^k}, \ k < \left\lceil \frac{\log r}{\log 2} \right\rceil. \tag{5.84}$$

Let $r = 10$, then only $k = 1, 2, 3$ satisfy the condition for convergence. Thus,

$$\sum_{n=0}^{\infty} \frac{\mathcal{J}_n}{10^{n+1}} = \frac{20-1}{100-10-2} = \frac{19}{88}, \tag{5.85}$$

$$\sum_{n=0}^{\infty} \frac{\mathcal{J}_{2n}}{10^{n+1}} = \frac{20-5}{100-50+4} = \frac{15}{54}, \tag{5.86}$$

$$\sum_{n=0}^{\infty} \frac{\mathcal{J}_{3n}}{10^{n+1}} = \frac{20-7}{100-70-8} = \frac{13}{22}. \tag{5.87}$$

5.4.4 Leonardo Sequence and Generalized Leonardo Sequence

In this section, we will present results involving the Leonardo sequence and the generalized Leonardo sequence mentioned in Section 5.3. Some definitions, theorems, and corollaries are repeated here for readability.

Definition 5.4.3. *(Catarino and Borges [11]) A second-order linear recursive sequence* $\{Le_n\}$ *is called the Leonardo sequence if it satisfies*

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \tag{5.88}$$

with $Le_0 = 1$ *and* $Le_1 = 1$.

Next, we will us the generalized Leonardo sequence $\{\mathcal{L}_{m,n}\}$ as follows:

Definition 5.4.4. *(Kuhapatanakul and Chobsorn [45]). The generalized Leonardo sequence* $\{\mathcal{L}_{m,n}\}$, *with a fixed positive integer* m, *is defined by*

$$\mathcal{L}_{m,n} = \mathcal{L}_{m,n-1} + \mathcal{L}_{m,n-2} + m, \ n \geq 2, \tag{5.89}$$

with the initial conditions $\mathcal{L}_{m,0} = \mathcal{L}_{m,1} = 1$.

Theorem 5.4.11. *(Kuhapatanakul and Chobsorn [45]). The closed formula for the generalized Leonardo sequence* $\{\mathcal{L}_{m,n}\}$ *is*

$$\mathcal{L}_{m,n} = (1+m)F_{n+1} - m. \tag{5.90}$$

Corollary 5.4.12. *(Catarino and Borges [11]). Let* $\{Le_n\}$ *be the classical Leonardo sequence be defined by* $Le_n = Le_{n-1} + Le_{n-2} + 1, \ n \geq 2$ *with initial conditions* $Le_0 = Le_1 = 1$. *Then*

$$Le_n = 2F_{n+1} - 1.$$

Theorem 5.4.13. *Let* $\{\mathcal{L}_{m,n}\}$ *be the generalized Leonardo sequence defined in Definition 5.3.1. Let* $\phi = \dfrac{1+\sqrt{5}}{2}$ *and* $\psi = \dfrac{1-\sqrt{5}}{2}$ *be two roots of the characteristic equation* $x^2 - x - 1 = 0$ *and* *satisfy*

$$k < \left\lceil \frac{\log r}{\log(\frac{1+\sqrt{5}}{2})} \right\rceil.$$

Then

$$\sum_{n=0}^{\infty} \frac{\mathcal{L}_{m,kn}}{r^{n+1}} = \frac{(1+m)}{\sqrt{5}} \left(\frac{\phi(r-\psi^k) - \psi(r-\phi^k)}{(r-\phi^k)(r-\psi^k)} \right) - \frac{m}{r-1}. \tag{5.91}$$

Proof. Using (5.90) and Theorem 5.4.8 yields

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\mathcal{L}_{m,kn}}{r^{n+1}} &= \sum_{n=0}^{\infty} \left(\frac{(1+m)F_{kn+1}}{r^{n+1}} - \frac{m}{r^{n+1}} \right) \\ &= (1+m) \sum_{n=0}^{\infty} \frac{F_{kn+1}}{r^{n+1}} - \frac{m}{r} \sum_{n=0}^{\infty} \left(\frac{1}{r} \right)^n \\ &= \frac{(1+m)}{\sqrt{5}} \left(\frac{\phi(r-\psi^k) - \psi(r-\phi^k)}{(r-\phi^k)(r-\psi^k)} \right) - \frac{m}{r-1}. \end{aligned}$$

□

Corollary 5.4.14. *Let $m = 1$ in Theorem 5.4.13, i.e., $\mathcal{L}_{1,kn} = Le_{kn}$. Then*

$$\sum_{n=0}^{\infty} \frac{Le_{kn}}{r^{n+1}} = \frac{2}{\sqrt{5}} \left(\frac{\phi(r-\psi^k) - \psi(r-\phi^k)}{(r-\phi^k)(r-\psi^k)} \right) - \frac{1}{r-1}, \tag{5.92}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$, and $k < \left\lceil \frac{\log r}{\log \phi} \right\rceil$.

Proof. Let $m = 1$ in (5.91), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\mathcal{L}_{1,kn}}{r^{n+1}} &= \sum_{n=0}^{\infty} \frac{Le_{kn}}{r^{n+1}} \\ &= \frac{2}{\sqrt{5}} \left(\frac{\phi(r-\psi^k) - \psi(r-\phi^k)}{(r-\phi^k)(r-\psi^k)} \right) - \frac{1}{r-1}, \end{aligned}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$, and $k < \left\lceil \frac{\log r}{\log \phi} \right\rceil$. □

Example 5.4.14. *Let $r = 10$ in (5.92), then only $k = 1, 2, 3, 4$ satisfy the condition for convergence.*

Thus,

$$\sum_{n=0}^{\infty}\frac{Le_n}{10^{n+1}} = \frac{2}{\sqrt{5}}\left(\frac{\phi(10-\psi)-\psi(10-\phi)}{(10-\phi)(10-\psi)}\right)-\frac{1}{10-1}=\frac{91}{801}, \tag{5.93}$$

$$\sum_{n=0}^{\infty}\frac{Le_{2n}}{10^{n+1}} = \frac{2}{\sqrt{5}}\left(\frac{\phi(10-\psi^2)-\psi(10-\phi^2)}{(10-\phi^2)(10-\psi^2)}\right)-\frac{1}{10-1}=\frac{91}{639}, \tag{5.94}$$

$$\sum_{n=0}^{\infty}\frac{Le_{3n}}{10^{n+1}} = \frac{2}{\sqrt{5}}\left(\frac{\phi(10-\psi^3)-\psi(10-\phi^3)}{(10-\phi^3)(10-\psi^3)}\right)-\frac{1}{10-1}=\frac{103}{531}, \tag{5.95}$$

$$\sum_{n=0}^{\infty}\frac{Le_{4n}}{10^{n+1}} = \frac{2}{\sqrt{5}}\left(\frac{\phi(10-\psi^4)-\psi(10-\phi^4)}{(10-\phi^4)(10-\psi^4)}\right)-\frac{1}{10-1}=\frac{113}{279}. \tag{5.96}$$

Next, we will consider a version of the Leonardo-like sequence $\{C_n(a,b,m)\}$ defined by

$$C_n(a,b,m) = C_{n-1}(a,b,m) + C_{n-2}(a,b,m) + m, \tag{5.97}$$

with $C_0(a,b,m) = b-a-m$, $C_1(a,b,m) = a$, and m is a constant (Bicknell-Johnson and Bergum [8]). The generalized Leonardo sequence arises as a special case of C_n :

$$\mathcal{L}_{m,n} = C_n(1, 2+m, m).$$

Lemma 5.4.3 (Bicknell-Johnson and Bergum [8])**.** . *Consider the Leonardo-like sequence* $\{C_n(a,b,m)\}$ *as defined in* (5.97)*. Then*

$$C_n(a,b,m) = aF_{n-2} + bF_{n-1} + m(F_n - 1).$$

The proof of this lemma can be found in (Bicknell-Johnson and Bergum [8]).

Theorem 5.4.15. *Let* $\{C_n(a,b,m)\}$ *be the Leonardo-like sequence. Then*

$$\sum_{n=0}^{\infty}\frac{C_{kn}(a,b,m)}{r^{n+1}} = \frac{(r-\phi^k)((b-a)\phi-m-a)-(r-\psi^k)((b-a)\psi-m-a)}{\sqrt{5}(r-\phi^k)(r-\psi^k)} - \frac{m}{r-1}.$$

Proof. Using Lemma 5.4.3, we have

$$\sum_{n=0}^{\infty}\frac{C_{kn}(a,b,m)}{r^{n+1}} = a\sum_{n=0}^{\infty}\frac{F_{kn-2}}{r^{n+1}} + b\sum_{n=0}^{\infty}\frac{F_{kn-1}}{r^{n+1}} + m\sum_{n=0}^{\infty}\frac{F_{kn}}{r^{n+1}} - m\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}.$$

We will deal with the series involving the Fibonacci sequence separately. Denote $\phi = \dfrac{1+\sqrt{5}}{2}$ and $\psi = \dfrac{1-\sqrt{5}}{2}$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{F_{kn-2}}{r^{n+1}} &= \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\left(\frac{\phi^{kn-2}-\psi^{kn-2}}{r^{n+1}}\right)\\
&= \frac{1}{r\phi^2\sqrt{5}}\sum_{n=0}^{\infty}\left(\frac{\phi^k}{r}\right)^n - \frac{1}{r\psi^2\sqrt{5}}\sum_{n=0}^{\infty}\left(\frac{\psi^k}{r}\right)^n\\
&= \frac{1}{r\phi^2\sqrt{5}}\left(\frac{1}{1-\frac{\phi^k}{r}}\right) - \frac{1}{r\psi^2\sqrt{5}}\left(\frac{1}{1-\frac{\psi^k}{r}}\right)\\
&= \frac{1}{\sqrt{5}}\left(\frac{\phi^{-2}(r-\psi^k)-\psi^{-2}(r-\phi^k)}{(r-\phi^k)(r-\psi^k)}\right).
\end{aligned}$$

Next, by Theorem 5.4.7, we have

$$\sum_{n=0}^{\infty}\frac{F_{kn-1}}{r^{n+1}} = \frac{1}{\sqrt{5}}\left(\frac{\phi(r-\phi^k)-\psi(r-\psi^k)}{(r-\phi^k)(r-\psi^k)}\right).$$

Next, by Theorem 5.4.1, we have

$$\sum_{n=0}^{\infty}\frac{F_{kn}}{r^{n+1}} = \frac{1}{\sqrt{5}}\left(\frac{\phi^k-\psi^k}{(r-\phi^k)(r-\psi^k)}\right).$$

Putting the above together, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{C_{kn}(a,b,m)}{r^{n+1}} &= a\sum_{n=0}^{\infty}\frac{F_{kn-2}}{r^{n+1}} + b\sum_{n=0}^{\infty}\frac{F_{kn-1}}{r^{n+1}} + m\sum_{n=0}^{\infty}\frac{F_{kn}}{r^{n+1}} - m\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\\
&= \frac{a}{\sqrt{5}}\left(\frac{\phi^{-2}(r-\psi^k)-\psi^{-2}(r-\phi^k)}{(r-\phi^k)(r-\psi^k)}\right) + \frac{b}{\sqrt{5}}\left(\frac{\phi(r-\phi^k)-\psi(r-\psi^k)}{(r-\phi^k)(r-\psi^k)}\right)\\
&\quad + \frac{m}{\sqrt{5}}\left(\frac{\phi^k-\psi^k}{(r-\phi^k)(r-\psi^k)}\right) - \frac{m}{r-1}\\
&= \frac{(r-\phi^k)(-a\psi^{-2}+b\phi-m)+(r-\psi_k)(a\phi^{-2}-b\psi+m)}{\sqrt{5}(r-\phi^k)(r-\psi^k)} - \frac{m}{r-1}
\end{aligned}$$

$$= \frac{(r-\phi^k)((b-a)\phi - m - a) - (r-\psi^k)((b-a)\psi - m - a)}{\sqrt{5}(r-\phi^k)(r-\psi^k)} - \frac{m}{r-1}.$$

□

We can generalize further by letting $\{w_n(w_0, w_1, p, q, t, j)\}$ be a sequence of order 2 satisfying the non-homogeneous linear relation in the following form (Shannon et al. [75]):

$$w_n = pw_{n-1} + qw_{n-2} + (p+q-1)(tn+j), \quad n \geq 2,\ t, j \in \mathbb{Z}, S \tag{5.98}$$

where w_0, w_1, p, q are given constants such that $p + q \neq 1$.

Lemma 5.4.4. *(Shannon et al. [75]). Let $\{w_n(w_0, w_1, p, q, t, j)\}$ be the sequence as defined in (5.98). Then*

$$w_n = w_n(w_0, w_1, p, q, 0, 0) + \left(j - \frac{t(p+2q)}{1-p-q}\right)(w_n(1,1,p,q,0,0) - 1) + t(w_n(0,1,p,q,0,0) - n).$$

Theorem 5.4.16. *Let $\{w_n(w_0, w_1, p, q, t, j)\}$ be the sequence as defined in (5.98). Then*

$$\sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}} = \frac{(r-p)w_0 + w_1 + (p+q-1)\left(\frac{j}{r-1} + \frac{t}{(r-1)^2}\right)}{r^2 - pr - q}. \tag{5.99}$$

, with $r > 1$.

Proof. Let S be the desired series, i.e.,

$$S = \sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}}.$$

Consider the straightforward calculations

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{w_{n+1}}{r^{n+1}} &= r\sum_{n=1}^{\infty} \frac{w_n}{r^{n+1}} = r\left(\sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}} - \frac{w_0}{r}\right) = r\sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}} - w_0, \\
\sum_{n=0}^{\infty} \frac{w_{n+2}}{r^{n+1}} &= r^2\sum_{n=2}^{\infty} \frac{w_n}{r^{n+1}} = r^2\left(\sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}} - \frac{w_1}{r^2} - \frac{w_0}{r}\right) = r^2\sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}} - w_1 - rw_0.
\end{aligned} \tag{5.100}$$

From (5.98), it is clear that

$$\sum_{n=0}^{\infty}\frac{w_{n+2}}{r^{n+1}}=\sum_{n=0}^{\infty}\frac{pw_{n+1}+qw_n+(p+q-1)(tn+j)}{r^{n+1}}$$
$$=p\sum_{n=0}^{\infty}\frac{w_{n+1}}{r^{n+1}}+q\sum_{n=0}^{\infty}\frac{w_n}{r^{n+1}}+t(p+q-1)\sum_{n=0}^{\infty}\frac{n}{r^{n+1}}+j(p+q-1)\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}.$$

Then by (5.100), we get

$$qS=\sum_{n=0}^{\infty}\frac{w_{n+2}}{r^{n+1}}-p\sum_{n=0}^{\infty}\frac{w_{n+1}}{r^{n+1}}-\frac{t(p+q-1)}{r}\sum_{n=0}^{\infty}n\left(\frac{1}{r}\right)^n-\frac{j(p+q-1)}{r}\sum_{n=0}^{\infty}\left(\frac{1}{r}\right)^n$$
$$=(Sr^2-w_0r-w_1)-p(Sr-w_0)-\frac{t(p+q-1)}{r}\cdot\frac{\frac{1}{r}}{\left(1-\frac{1}{r}\right)^2}-\frac{j(p+q-1)}{r}\cdot\frac{1}{1-\frac{1}{r}}$$
$$=S(r^2-pr)-w_0r-pw_0-w_1-\frac{t(p+q-1)}{(r-1)^2}-\frac{j(p+q-1)}{r-1}.$$

Solving this equation for S yields the desired equality. □

5.4.5 Linear Recurrence Sequences of Order Three

We now analyze the natural corresponding results for sequences of order three. First and foremost, we derive a closed form expression for third order linear recurrence sequences. A matrix representation for the closed form can be found in He et al. [28].

Theorem 5.4.17. *Let $\{a_n\}$ be a sequence satisfying the third order recurrence relation*

$$a_n=pa_{n-1}+qa_{n-2}+ta_{n-3},\quad n\geq 3,\tag{5.101}$$

for some constants $p,q,t\neq 0$ and initial conditions $a_0,a_1,$ and a_2. Let α,β,γ be roots of the characteristic equation $x^3-px^2-qx-t=0$. Then we have the following third order analogue of Theorem 5.3.4:

$$a_n=\begin{cases}\dfrac{a_1(\beta+\gamma)-a_0\beta\gamma-a_2}{(\alpha-\beta)(\gamma-\alpha)}\alpha^n+\dfrac{a_1(\alpha+\gamma)-a_0\alpha\gamma-a_2}{(\alpha-\beta)(\beta-\gamma)}\beta^n+\dfrac{a_1(\alpha+\beta)-a_0\alpha\beta-a_2}{(\beta-\gamma)(\gamma-\alpha)}\gamma^n, & \text{if } \alpha\neq\beta\neq\gamma,\\ \dfrac{2a_1-2a_0\gamma}{(\gamma-\alpha)^2}\alpha^{n+1}+\dfrac{a_0\gamma^2-a_2}{(\gamma-\alpha)^2}\alpha^n+\dfrac{a_1(\gamma+\alpha)-a_0\alpha\gamma-a_2}{\gamma-\alpha}n\alpha^{n-1}+\dfrac{a_0\alpha^2-2a_1\alpha+a_2}{(\gamma-\alpha)^2}\gamma^n, & \text{if } \alpha=\beta\neq\gamma,\\ \dfrac{1}{2}\left[a_0(n-1)(n-2)\alpha^n-2a_1n(n-2)\alpha^{n-1}+a_2n(n-1)\alpha^{n-2}\right], & \text{if } \alpha=\beta=\gamma.\end{cases}\tag{5.102}$$

Proof. First, suppose that α, β, γ are all distinct. By Vieta's formula, we have

$$\begin{aligned} p &= \alpha + \beta + \gamma \\ q &= -(\alpha\beta + \beta\gamma + \gamma\alpha) \\ r &= \alpha\beta\gamma. \end{aligned}$$

Substituting these into (5.101) yields

$$a_n = (\alpha + \beta + \gamma)a_{n-1} - (\alpha\beta + \beta\gamma + \gamma\alpha)a_{n-2} + \alpha\beta\gamma a_{n-3}.$$

This is equivalent to

$$a_n - (\alpha + \beta)a_{n-1} + \alpha\beta a_{n-2} = \gamma(a_{n-1} - (\alpha + \beta)a_{n-2} + \alpha\beta a_{n-3}),$$

which implies that the sequence $\{a_n - (\alpha+\beta)a_{n-1} + \alpha\beta a_{n-2}\}$ is geometric, with γ being its common ratio. Thus,

$$a_n - (\alpha + \beta)a_{n-1} + \alpha\beta a_{n-2} = (a_2 - (\alpha + \beta)a_1 + \alpha\beta a_0)\gamma^{n-2}$$

which implies

$$\frac{a_n}{\gamma^n} = \frac{\alpha + \beta}{\gamma} \cdot \frac{a_{n-1}}{\gamma^{n-1}} - \frac{\alpha\beta}{\gamma^2} \cdot \frac{a_{n-2}}{\gamma^{n-2}} + \frac{a_2 - (\alpha + \beta)a_1 + \alpha\beta a_0}{\gamma^2}.$$

We can make the substitution $A_n := a_n/\gamma^n$ to yield the second-order nonhomogeneous recurrence relation

$$A_n = \frac{\alpha + \beta}{\gamma} A_{n-1} - \frac{\alpha\beta}{\gamma^2} A_{n-2} + \frac{a_2 - (\alpha + \beta)a_1 + \alpha\beta a_0}{\gamma^2}, \tag{5.103}$$

where $A_0 = a_0$ and $A_1 = a_1/\gamma$.

Solving the corresponding characteristic equation

$$x^2 - \frac{\alpha+\beta}{\gamma}x + \frac{\alpha\beta}{\gamma^2} = \frac{a_2 - (\alpha+\beta)a_1 + \alpha\beta a_0}{\gamma^2},$$

we get the roots $x = \dfrac{\alpha}{\gamma}, \dfrac{\beta}{\gamma}$. This implies that the closed form for $\{a_n\}$ is of the form

$$A_n = c_1\left(\frac{\alpha}{\gamma}\right)^n + c_2\left(\frac{\beta}{\gamma}\right)^n + C, \tag{5.104}$$

with undetermined coefficients c_1, c_2, and C.

With C being a particular solution of A_n, it satisfies (5.103), and so

$$C = \frac{\alpha+\beta}{\gamma}\cdot C - \frac{\alpha\beta}{\gamma^2}\cdot C + \frac{a_2 - (\alpha+\beta)a_1 + \alpha\beta a_0}{\gamma^2}$$

which implies

$$C = \frac{a_1(\alpha+\beta) - a_0\alpha\beta - a_2}{(\beta-\gamma)(\gamma-\alpha)}.$$

Evaluating (5.103) at $n = 0$ and $n = 1$ yields

$$\begin{aligned} a_0 &= A_0 = c_1 + c_2 + C, \\ \frac{a_1}{\gamma} &= A_1 = c_1\frac{\alpha}{\gamma} + c_2\frac{\beta}{\gamma} + C, \end{aligned}$$

respectively. This system of two equations with two unknowns c_1 and c_2 can be solved to get

$$\begin{aligned} c_1 &= \frac{a_1(\beta+\gamma) - a_0\beta\gamma - a_2}{(\alpha-\beta)(\gamma-\alpha)}, \\ c_2 &= \frac{a_1(\alpha+\gamma) - a_0\alpha\gamma - a_2}{(\alpha-\beta)(\beta-\gamma)}. \end{aligned}$$

Thus, multiplying both sides of (5.103) by γ^n gifts us with

$$a_n = \frac{a_1(\beta+\gamma) - a_0\beta\gamma - a_2}{(\alpha-\beta)(\gamma-\alpha)}\alpha^n + \frac{a_1(\alpha+\gamma) - a_0\alpha\gamma - a_2}{(\alpha-\beta)(\beta-\gamma)}\beta^n + \frac{a_1(\alpha+\beta) - a_0\alpha\beta - a_2}{(\beta-\gamma)(\gamma-\alpha)}\gamma^n. \tag{5.105}$$

Now, supposing $\alpha = \beta \neq \gamma$, we can simply let β approach α in (5.105). Starting with the third term of the right-hand side of (5.105), we get

$$\lim_{\beta\to\alpha} \frac{a_1(\alpha+\beta) - a_0\alpha\beta - a_2}{(\beta-\gamma)(\gamma-\alpha)}\gamma^n = \frac{a_0\alpha^2 - 2a_1\alpha + a_2}{(\gamma-\alpha)^2}\gamma^n. \tag{5.106}$$

Treating the first two terms of the the right-hand side of (5.105) carefully,

$$\begin{aligned}
&\lim_{\beta\to\alpha}\left[\frac{a_1(\beta+\gamma) - a_0\beta\gamma - a_2}{(\alpha-\beta)(\gamma-\alpha)}\alpha^n + \frac{a_1(\alpha+\gamma) - a_0\alpha\gamma - a_2}{(\alpha-\beta)(\beta-\gamma)}\beta^n\right] \\
=&\lim_{\beta\to\alpha}\left[\frac{a_1(\beta^2-\gamma^2) - a_0\beta\gamma(\beta-\gamma) - a_2(\beta-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}\alpha^n + \frac{a_1(\gamma^2-\alpha^2) - a_0\alpha\gamma(\gamma-\alpha) - a_2(\gamma-\alpha)}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}\beta^n\right],
\end{aligned}$$

we employ L'Hôpital rule to continue to

$$\begin{aligned}
&= \lim_{\beta\to\alpha}\left[\frac{2a_1\beta - 2a_0\beta\gamma + a_0\gamma^2 - a_2}{(\alpha-2\beta+\gamma)(\gamma-\alpha)}\alpha^n + \frac{na_1(\gamma^2-\alpha^2) - a_0\alpha\gamma(\gamma-\alpha) - a_2(\gamma-\alpha)}{(\alpha-2\beta+\gamma)(\gamma-\alpha)}\beta^{n-1}\right] \\
&= \frac{2a_1 - 2a_0}{(\gamma-\alpha)^2}\alpha^{n+1} + \frac{a_0\gamma^2 - a_2}{(\gamma-\alpha)^2}\alpha^n + \frac{n\alpha^{n-1}\left(a_1(\gamma+\alpha) - a_0\alpha\gamma - a_2\right)}{\gamma-\alpha}.
\end{aligned} \tag{5.107}$$

Combining (5.106) and (5.107), we obtain the formula for a_n when $\alpha = \beta \neq \gamma$:

$$a_n = \frac{2a_1 - 2a_0\gamma}{(\gamma-\alpha)^2}\alpha^{n+1} + \frac{a_0\gamma^2 - a_2}{(\gamma-\alpha)^2}\alpha^n + \frac{a_1(\gamma+\alpha) - a_0\alpha\gamma - a_2}{\gamma-\alpha}n\alpha^{n-1} + \frac{a_0\alpha^2 - 2a_1\alpha + a_2}{(\gamma-\alpha)^2}\gamma^n \tag{5.108}$$

Finally, to get the case $\alpha = \beta = \gamma$, we let γ approach α in (5.108). Indeed,

$$\begin{aligned}
a_n = \lim_{\gamma\to\alpha}\frac{1}{(\gamma-\alpha)^2}\Big[&(2a_1 - 2a_0\gamma)\alpha^{n+1} + (a_0\gamma^2 - a_2)\alpha^n \\
&+ n\alpha^{n-1}(a_1\gamma + a_1\alpha - a_0\alpha\gamma - a_2)(\gamma-\alpha) + (a_0\alpha^2 - 2a_1\alpha + a_2)\gamma^n\Big]
\end{aligned}$$

Apply L'Hôpital rule once, we have

$$\begin{aligned}
a_n = \lim_{\gamma\to\alpha}\frac{1}{2(\gamma-\alpha)}\Big[&-2a_0\alpha^{n+1} + 2a_0\gamma\alpha^n + n\gamma^{n-1}(a_0\alpha^2 - 2a_1\alpha + a_2) \\
&+ n\alpha^{n-1}(2a_1\gamma - 2a_0\alpha\gamma - a_2 + a_0\alpha^2)\Big].
\end{aligned}$$

Apply L'Hôpital rule again and rearranging terms, we have the result for $\alpha = \beta = \gamma$:

$$a_n = \frac{1}{2}\Big[a_0(n-1)(n-2)\alpha^n - 2a_1 n(n-2)\alpha^{n-1} + a_2 n(n-1)\alpha^{n-2}\Big].$$

□

Next, we consider criteria for convergence of the usual series. To do so, we introduce the following lemma:

Lemma 5.4.5. *Suppose* $|t| < 1$. *Then*

$$\sum_{n=0}^{\infty} n^2 t^n = \frac{t^2 + t}{(1-t)^3}.$$

Proof. Differentiating the series in Lemma 5.4.2 with respect to t, we have

$$\sum_{n=1}^{\infty} n^2 t^{n-1} = \frac{1-t^2}{(1-t)^4} = \frac{t+1}{(1-t)^3}.$$

Multiplying by t yields the desired result. □

Lemma 5.4.5 serves as the final stepping stone needed for the following result on convergence.

Lemma 5.4.6. *Let* $\{a_n\}$ *be a sequence as defined in 5.4.17. Let* α, β, γ *be the roots of its characteristic equation that may or may not be distinct. Then the series*

$$\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}$$

converges if $r > \max\{|\alpha|, |\beta|, |\gamma|\}$.

Proof. Considering the first branch of (5.102), i.e., when α, β, γ are distinct, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} &= \sum_{n=0}^{\infty} \frac{a_1(\beta+\gamma) - a_0\beta\gamma - a_2}{(\alpha-\beta)(\gamma-\alpha)} \frac{\alpha^n}{r^{n+1}} + \sum_{n=0}^{\infty} \frac{a_1(\alpha+\gamma) - a_0\alpha\gamma - a_2}{(\alpha-\beta)(\beta-\gamma)} \frac{\beta^n}{r^{n+1}} \\ &\quad + \frac{a_1(\alpha+\beta) - a_0\alpha\beta - a_2}{(\beta-\gamma)(\gamma-\alpha)} \frac{\gamma^n}{r^{n+1}} \\ &= \frac{a_1(\beta+\gamma) - a_0\beta\gamma - a_2}{r(\alpha-\beta)(\gamma-\alpha)} \sum_{n=0}^{\infty} \left(\frac{\alpha}{r}\right)^n + \frac{a_1(\alpha+\gamma) - a_0\alpha\gamma - a_2}{r(\alpha-\beta)(\beta-\gamma)} \sum_{n=0}^{\infty} \left(\frac{\beta}{r}\right)^n\end{aligned}$$

$$+\frac{a_1(\alpha+\gamma)-a_0\alpha\gamma-a_2}{r(\alpha-\beta)(\beta-\gamma)}\sum_{n=0}^{\infty}\left(\frac{\gamma}{r}\right)^n,$$

where the geometric series in the above line converge when $\frac{|\alpha|}{r}, \frac{|\beta|}{r}, \frac{|\gamma|}{r} < 1$. Or, equivalently, when $r > \max\{|\alpha|, |\beta|, |\gamma|\}$.

As for the second branch of (5.102), another direct computation foretells

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{a_n}{r^{n+1}} &= \sum_{n=0}^{\infty}\frac{2a_1-2a_0\gamma}{(\gamma-\alpha)^2}\frac{\alpha^{n+1}}{r^{n+1}} + \sum_{n=0}^{\infty}\frac{a_0\gamma^2-a_2}{(\gamma-\alpha)^2}\frac{\alpha^n}{r^{n+1}} + \sum_{n=0}^{\infty}\frac{a_1(\gamma+\alpha)-a_0\alpha\gamma-a_2}{\gamma-\alpha}\cdot n\frac{\alpha^{n-1}}{r^{n+1}} \\
&\quad + \sum_{n=0}^{\infty}\frac{a_0\alpha^2-2a_1\alpha+a_2}{(\gamma-\alpha)^2}\frac{\gamma^n}{r^{n+1}} \\
&= \frac{2a_1-2a_0\gamma}{(\gamma-\alpha)^2}\cdot\frac{\alpha}{r}\sum_{n=0}^{\infty}\left(\frac{\alpha}{r}\right)^n + \frac{a_0\gamma^2-a_2}{r(\gamma-\alpha)^2}\sum_{n=0}^{\infty}\left(\frac{\alpha}{r}\right)^n \\
&\quad + \frac{a_1(\gamma+\alpha)-a_0\alpha\gamma-a_2}{\alpha r(\gamma-\alpha)}\sum_{n=0}^{\infty}n\left(\frac{\alpha}{r}\right)^n + \frac{a_0\alpha^2-2a_1\alpha+a_2}{r(\gamma-\alpha)^2}\sum_{n=0}^{\infty}\left(\frac{\gamma}{r}\right)^n
\end{aligned}$$

where the same condition elicits convergence of the above series.

Finally, the last branch results in

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{a_n}{r^{n+1}} &= \frac{1}{2}\left[\sum_{n=0}^{\infty}a_0(n-1)(n-2)\frac{\alpha^n}{r^{n+1}} - \sum_{n=0}^{\infty}2a_1n(n-2)\frac{\alpha^{n-1}}{r^{n+1}} + \sum_{n=0}^{\infty}a_2n(n-1)\frac{\alpha^{n-2}}{r^{n+1}}\right] \\
&= \frac{a_0}{2r}\left[\sum_{n=0}^{\infty}n^2\left(\frac{\alpha}{r}\right)^n - 3\sum_{n=0}^{\infty}n\left(\frac{\alpha}{r}\right)^n + 2\sum_{n=0}^{\infty}\left(\frac{\alpha}{r}\right)^n\right] - \frac{a_1}{\alpha r}\left[\sum_{n=0}^{\infty}n^2\left(\frac{\alpha}{r}\right)^n - 2\sum_{n=0}^{\infty}n\left(\frac{\alpha}{r}\right)^n\right] \\
&\quad + \frac{a_2}{2\alpha^2 r}\left[\sum_{n=0}^{\infty}n^2\left(\frac{\alpha}{r}\right)^n - \sum_{n=0}^{\infty}n\left(\frac{\alpha}{r}\right)^n\right],
\end{aligned}$$

where a special appearance of Lemma 5.4.5 guarantees convergence with the same condition. □

Behold, the main theorem of this section.

Theorem 5.4.18. *Let $\{a_n\}$ be a sequence satisfying the recurrence relation in (5.101) and let $r > \max\{|\alpha|, |\beta|, |\gamma|, 1\}$. Then*

$$\sum_{n=0}^{\infty}\frac{a_n}{r^{n+1}} = \frac{a_0r^2+(a_1-pa_0)r+(a_2-pa_1-qa_0)}{r^3-pr^2-qr-t}.$$

Proof. Denote the series to be evaluated by S, i.e.,

$$S = \sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}.$$

We first note that

$$\sum_{n=0}^{\infty} \frac{a_{n+3}}{r^{n+1}} = \sum_{n=0}^{\infty} \frac{pa_{n+2} + qa_{n+1} + ta_n}{r^{n+1}} = p\sum_{n=0}^{\infty} \frac{a_{n+2}}{r^{n+1}} + q\sum_{n=0}^{\infty} \frac{a_{n+1}}{r^{n+1}} + t\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}. \tag{5.109}$$

Since we know from Lemma 5.4.6 that S converges, we can use the usual process to yield the equalities:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{a_{n+1}}{r^{n+1}} &= r\sum_{n=1}^{\infty} \frac{a_n}{r^{n+1}} = r\left(\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - \frac{a_0}{r}\right) = Sr - a_0, \\
\sum_{n=0}^{\infty} \frac{a_{n+2}}{r^{n+1}} &= r^2\sum_{n=2}^{\infty} \frac{a_n}{r^{n+1}} = r^2\left(\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - \frac{a_0}{r} - \frac{a_1}{r^2}\right) = Sr^2 - a_0 r - a_1, \\
\sum_{n=0}^{\infty} \frac{a_{n+3}}{r^{n+1}} &= r^3\sum_{n=3}^{\infty} \frac{a_n}{r^{n+1}} = r^3\left(\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - \frac{a_0}{r} - \frac{a_1}{r^2} - \frac{a_2}{r^3}\right) = Sr^3 - a_0 r^2 - a_1 r - a_2.
\end{aligned}$$

Combining these in (5.109), we get

$$\begin{aligned}
tS &= (Sr^3 - a_0 r^2 - a_1 r - a_2) - p(Sr^2 - a_0 r - a_1) - q(Sr - a_0) \\
&= S(r^3 - pr^2 - qr) - a_0 r^2 + (pa_0 - a_1)r + (qa_0 + pa_1 - a_2).
\end{aligned}$$

Upon solving for S, the desired result arises. □

Example 5.4.15. *The Padovan sequence $\{p_n\}$ is defined by the recurrence relation*

$$p_n = p_{n-2} + p_{n-3}, \quad n \geq 3,$$

with $p_0 = p_1 = p_2 = 1$. The real root to the characteristic equation $x^3 - x - 1 = 0$ is $\alpha \approx 1.3247$ (Plastic ratio, Shannon et al. [72]). Hence, for any $r > 1.3247$, we have

$$\sum_{n=0}^{\infty} \frac{p_n}{r^{n+1}} = \frac{r^2 + r}{r^3 - r - 1}, \tag{5.110}$$

and in particular,

$$\sum_{n=0}^{\infty} \frac{p_n}{10^{n+1}} = \frac{110}{989} \quad \textit{and} \quad \sum_{n=0}^{\infty} \frac{p_n}{2^{n+1}} = \frac{24}{5}. \tag{5.111}$$

Example 5.4.16. *The Perrin sequence $\{q_n\}$ is defined by the recurrence relation*

$$q_n = q_{n-2} + q_{n-3}, \quad n \geq 3,$$

with the initial conditions $q_0 = 3, q_1 = 0, q_2 = 2$. For $r > 1.3247$, we have

$$\sum_{n=0}^{\infty} \frac{q_n}{r^{n+1}} = \frac{3r^2 - 1}{(r-\alpha)(r-\beta)(r-\gamma)} = \frac{3r^2 - 1}{r^3 - r - 1}.$$

Example 5.4.17. *A third-order linear recursive sequence $\{\mathcal{T}_n\}$ is called a Tribonacci sequence if it satisfies*

$$\mathcal{T}_n = \mathcal{T}_{n-1} + \mathcal{T}_{n-2} + \mathcal{T}_{n-3}, \ n \geq 3,$$

with $\mathcal{T}_0 = 0, \ \mathcal{T}_1 = 1, \ \mathcal{T}_2 = 1$. The real root to the characteristic equation $x^3 - x^2 - x - 1 = 0$ is $\alpha \approx 1.8393$ (Tribonacci constant, Hudson [37]). Hence, for any $r > 1.8393$, we have

$$\sum_{n=0}^{\infty} \frac{\mathcal{T}_n}{r^{n+1}} = \frac{r}{r^3 - r^2 - 2},$$

and particularly,

$$\sum_{n=0}^{\infty} \frac{\mathcal{T}_n}{10^{n+1}} = \frac{5}{449} \quad \textit{and} \quad \sum_{n=0}^{\infty} \frac{\mathcal{T}_n}{2^{n+1}} = 1. \tag{5.112}$$

The series for the Tribonacci sequence was also discussed by Hudson [37].

5.4.6 Series Value and Approximation

Example 5.4.18. *Recall the sequence of Mersenne numbers $\{M_n\}$ from Example 5.4.8 defined by*

$$M_n = 3M_{n-1} - 2M_{n-2}, n \geq 2,$$

with $M_0 = 0$ and $M_1 = 1$. Consider the speed of convergence of the partial sum

$$\sum_{n=1}^{N} \frac{M_n}{10^{n+1}}.$$

From (5.63), this partial sum converges to $\frac{1}{72} = 0.013\bar{8}$ as $N \to \infty$. Note that if $N = 4$, the partial sum evaluates to 0.01385; if $N = 5$, it evaluates to 0.013881; if $N = 6$, we get 0.0138873; if $N = 7$, we get 0.01388857, and so on. In particular, for $N \geq 4$, the partial sum dominates 0.0138, implying that the convergence is not slow.

Definition 5.4.5. *We will say that two rational numbers with the repeating digits*

$$p_1 p_2 \cdots p_n,$$

$$q_1 q_2 \cdots q_m,$$

respectively, are called cyclic numbers if $m = n$ and there exists an integer k such that $0 \leq k < n$ and

$$p_i = q_{i+k}, \quad 0 \leq i < n.$$

Two numbers being cyclic are also commonly referred to as being in the same family.

Lemma 5.4.7. *The distinct families of rational numbers of the form $\frac{a}{89}$, where $0 < a < 89$, are*

$$\mathcal{F}_1 = \left\{ \frac{1}{89}, \frac{2}{89}, \frac{4}{89}, \frac{5}{89}, \frac{8}{89}, \frac{9}{89}, \frac{10}{89}, \frac{11}{89}, \frac{16}{89}, \frac{17}{89}, \frac{18}{89}, \frac{20}{89}, \frac{21}{89}, \frac{22}{89}, \frac{25}{89}, \frac{32}{89}, \frac{34}{89}, \frac{36}{89}, \frac{39}{89}, \frac{40}{89}, \frac{42}{89}, \frac{44}{89}, \right.$$
$$\left. \frac{45}{89}, \frac{47}{89}, \frac{49}{89}, \frac{50}{89}, \frac{53}{89}, \frac{55}{89}, \frac{57}{89}, \frac{64}{89}, \frac{67}{89}, \frac{68}{89}, \frac{69}{89}, \frac{71}{89}, \frac{72}{89}, \frac{73}{89}, \frac{78}{89}, \frac{79}{89}, \frac{80}{89}, \frac{81}{89}, \frac{84}{89}, \frac{85}{89}, \frac{87}{89}, \frac{88}{89} \right\},$$
$$\mathcal{F}_2 = \left\{ \frac{3}{89}, \frac{6}{89}, \frac{7}{89}, \frac{12}{89}, \frac{13}{89}, \frac{14}{89}, \frac{15}{89}, \frac{19}{89}, \frac{23}{89}, \frac{24}{89}, \frac{26}{89}, \frac{27}{89}, \frac{28}{89}, \frac{29}{89}, \frac{30}{89}, \frac{31}{89}, \frac{33}{89}, \frac{35}{89}, \frac{37}{89}, \frac{38}{89}, \frac{41}{89}, \frac{43}{89}, \right.$$

$$\frac{46}{89}, \frac{48}{89}, \frac{51}{89}, \frac{52}{89}, \frac{54}{89}, \frac{56}{89}, \frac{58}{89}, \frac{59}{89}, \frac{60}{89}, \frac{61}{89}, \frac{62}{89}, \frac{63}{89}, \frac{65}{89}, \frac{66}{89}, \frac{70}{89}, \frac{74}{89}, \frac{75}{89}, \frac{76}{89}, \frac{77}{89}, \frac{82}{89}, \frac{83}{89}, \frac{86}{89}\bigg\},$$

where

$$\frac{1}{89} = 0.\overline{01123595505617977528089887640449438202247191},$$
$$\frac{3}{89} = 0.\overline{03370786516853932584269662921348314606741573}.$$

The following theorem showcases how these two families further solidify the importance of $\frac{1}{89}$.

Theorem 5.4.19. *Let $\mathcal{F}_1, \mathcal{F}_2$ be the families defined as in Lemma 5.4.7. Consider the sequences*

$$\mathcal{S}_1 = \{a_n\}_{n\geq 0} = \left\{\frac{10^n \pmod{89}}{89}\right\}_{n\geq 0} = \left\{\frac{1}{89}, \frac{10}{89}, \frac{11}{89}, \frac{21}{89}, \frac{32}{89}, \dots, \frac{17}{89}, \frac{81}{89}, \frac{9}{89}\right\},$$
$$\mathcal{S}_2 = \{b_n\}_{n\geq 0} = \left\{\frac{3\cdot 10^n \pmod{89}}{89}\right\}_{n\geq 0} = \left\{\frac{3}{89}, \frac{30}{89}, \frac{33}{89}, \frac{63}{89}, \frac{7}{89}, \dots, \frac{51}{89}, \frac{65}{89}, \frac{27}{89}\right\}.$$

Then $\mathcal{F}_1 = \mathcal{S}_1, \mathcal{F}_2 = \mathcal{S}_2$, and $\{a_n\}$ and $\{b_n\}$ satisfy the second order linear recurrence sequences

$$a_n = a_{n-1} \oplus a_{n-2}, \quad n \geq 2,$$

$$b_n = b_{n-1} \oplus b_{n-2}, \quad n \geq 2,$$

with $a_0 = \frac{1}{89}, a_1 = \frac{10}{89}, b_0 = \frac{3}{89}, b_1 = \frac{30}{89}$, where $\frac{a}{89} \oplus \frac{b}{89} := \frac{(a+b) \pmod{89}}{89}$.

Proof. Note that by definition of the sequence $\mathcal{S}_1$, we have

$$\begin{aligned} a_{n+2} &= \frac{10^{n+2} \pmod{89}}{89} \\ &= \frac{11\cdot 10^n \pmod{89}}{89} \\ &= \frac{(10+1)10^n \pmod{89}}{89} \\ &= \frac{10^{n+1} \pmod{89}}{89} + \frac{10^n \pmod{89}}{89} \\ &= a_{n+1} \oplus a_n. \end{aligned}$$

The same process yields the relation for $\{b_n\}$. □

Theorem 5.4.20. *Let* $\frac{1}{m} = (r_0 r_1 \cdots r_t)_{10}$ *and let* b *be a positive integer such that* $b \equiv 10^k \pmod{m}$*. Then*

$$\frac{b}{m} = (r_k r_{k+1} \cdots r_0 r_1 \cdots r_{k-1})_{10}.$$

That is, $\frac{1}{m}$ *and* $\frac{b}{m}$ *are cyclic numbers with a shift depending on* b*.*

For the proof of Theorem 5.4.20, please see Childs [16].

Example 5.4.19. *Let* $m = 89$*. Then letting* $b \equiv 10^k \pmod{89}$ *and letting* k *run from 1 until the process repeats, we obtain*

$$\begin{aligned}
\frac{1}{89} &= (011235955 \cdots 91)_{10}, \quad b^0 \equiv 10^0 \equiv 1 \pmod{89} \\
\frac{10}{89} &= (11235955 \cdots 910)_{10}, \quad b^1 \equiv 10^1 \equiv 10 \pmod{89}, \\
\frac{11}{89} &= (1235955 \cdots 9101)_{10}, \quad b^2 \equiv 100 \equiv 11 \pmod{89}, \\
\frac{21}{89} &= (235955 \cdots 91011)_{10}, \quad b^3 \equiv 110 \equiv 21 \pmod{89}, \\
\frac{32}{89} &= (35955 \cdots 910112)_{10}, \quad b^4 \equiv 210 \equiv 32 \pmod{89}, \\
\frac{53}{89} &= (5955 \cdots 9101123)_{10}, \quad b^5 \equiv 320 \equiv 53 \pmod{89}, \\
\frac{85}{89} &= (955 \cdots 91011235)_{10}, \quad b^6 \equiv 530 \equiv 85 \pmod{89}, \\
\frac{49}{89} &= (55 \cdots 910112359)_{10}, \quad b^7 \equiv 850 \equiv 49 \pmod{89} \\
&\vdots \\
\frac{81}{89} &= (910 \cdots 82022471)_{10}, \quad b^{42} \equiv 81 \pmod{89} \\
\frac{9}{89} &= (10 \cdots 820224719)_{10}, \quad b^{43} \equiv 9 \pmod{89}.
\end{aligned}$$

5.5 Conclusion

Jarden [38] has also considered Leonardo sequences from the point of view of the following variation of the Leonardo equation related to equation (5.3):

$$a_n = a_{n-1} + a_{n-2} \mp 1, \; n \geq 2, \tag{5.113}$$

and the associated third order linear recurrence

$$b_n = 2b_{n-1} - b_{n-3},\ n \geq 3, \tag{5.114}$$

to which the Leonardo sequences conform as in equation (5.3) with $k = \mp 1$. In fact, Jarden considers the sequences in Tables 5.3, 5.4, and 5.5, which can bring out the corresponding relations with the Fibonacci and Lucas sequences. $\{u_n\}$ is the sequence of differences, and is related to the generalized Fibonacci numbers of Jarden [38] in Table 5.10 and the hyper-Fibonacci and hyper-Lucas numbers in Table 5.11 (Dil and Mező [20]) with further generalized and extended Leonardo numbers.

(−1)	0	1	2	3	4	5	6	7	8
U_n	1	2	2	3	4	6	9	14	22
V_n	3	2	4	5	8	12	19	30	48
(+1)	0	1	2	3	4	5	6	7	8
U_n	−1	0	0	1	2	4	7	12	20
V_n	1	0	2	3	6	10	17	28	46

Table 5.10: Jarden's example of equation (5.3) with $k = \mp 1$.

Table 5.11 below is copied from Table 1 Alp [2]. It shows the interested reader the salient features of these sequences, both horizontally and vertically, as well as diagonally. Further properties to be investigated include intersections between sequences (Horadam [31]) and step functions within sequences (Chu et al. [17]). The last of these leads to s-Pascal triangles, as in Table 5.12.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	$\cdots$
$F_n^{(0)}$	0	1	1	2	3	5	8	13	21	34	$\cdots$
$L_n^{(0)}$	2	1	3	4	7	11	18	29	47	76	$\cdots$
$F_n^{(1)}$	0	1	2	4	7	12	20	33	54	88	$\cdots$
$L_n^{(1)}$	2	3	6	10	17	28	46	75	122	198	$\cdots$
$F_n^{(2)}$	0	1	3	7	14	26	46	79	133	221	$\cdots$
$L_n^{(2)}$	2	5	11	21	38	66	112	187	309	507	$\cdots$
$F_n^{(3)}$	0	1	4	11	25	51	97	176	309	530	$\cdots$
$L_n^{(3)}$	2	7	18	39	77	143	225	442	751	1258	$\cdots$

Table 5.11: Hyper-Fibonacci and hyper-Lucas numbers.

1													**1**
1	1	1											**3**
1	2	3	2	1									**9**
1	3	6	7	6	3	1							**27**
1	4	10	16	19	16	10	4	1					**81**
1	5	15	30	45	51	45	30	15	5	1			**243**
1	6	21	50	90	126	141	126	90	50	21	6	1	**729**

Table 5.12: A simple s-Pascal triangle.

If we then add along the leading diagonals in Table 5.12, we seem to arrive at the Tribonacci numbers, which can generate third-order Leonardo numbers.

In a different, but somewhat similar manner, Lind [52] defined $L(n, r)$ the r-th order nonlinear binomial sum as the sum of the first r terms of the $(n-1)$-th row of the ordinary Pascal's triangle plus the terms of the rising stair-step (or rising) diagonal originating at the r-th term, which can be applied to any of these tables. For example, in Table 5.12, we can have

$$L(1,3) = 1,\ L(2,3) = 3,\ L(3,3) = 6,\ L(4,3) = 12,\ L(4,4) = 18.$$

All of these can provide a nexus between the numerical results in this paper and the recent combinatorial work of Shattuck [76], who provided a framework for these and other identities satisfied by the Leonardo numbers in the notation of section 5.3 and other generalized and extended Fibonacci numbers. The initial step in extending Corollary 5.3.19 is

$$w_n = w_{n-1} + w_{n-2} + tn + j,\ n \geq 2,\ j > -4,$$

and

$$w_n = w_{n-1} + F_{n+1} - 1. \tag{5.115}$$

One can then extend the process to other second order sequences (Ollerton and Shannon [64]) or to other orders and other dimensions (Shannon [71]) for further related combinatorial properties. In this way, one can relate

$$w_n = w_{n-1} + w_{n-2} + tn + j,\ n \geq 2,\ t \geq 1,$$

and

$$w_n = w_{n-1} + F_n^{[k]}, \tag{5.116}$$

in which $F_n^{[k]}$ is hyper-Fibonacci sequence, as in Table 5.11, the rows of which as k increases can be seen as staked on top of one another for a third dimension. These can be developed further (Bahşi and Solak [5]). We note the neat recurrence relation

$$F_n^{[k]} = F_{n-1}^{[k]} + F_n^{[k-1]},\ k, n > 0, \tag{5.117}$$

with boundary conditions $F_n^{[0]} = F_n$ and $F_0^{[k]} = 0$; and with an elegant characteristic polynomial

$$(x^2 - x - 1)(x - 1)^k,$$

so that

$$F_n^{[k]} = \sum_{j=1}^{n} \binom{k+n-j-1}{k-1} F_j; \tag{5.118}$$

see Komatsu and Szalay [43] for details, including their relation to the infinite matrix in which $F_n^{[k]}$ is the entry in the n-th row and k-th column, and from there to Stirling numbers of the first kind.

In addition, this chapter has delved into the fascinating realm of infinite series, particularly focusing on sequences that generalize the famous Fibonacci sequences and their extensions. By referencing the seminal work of Melham and Shannon [58] from three decades ago, the chapter establishes a foundation for its exploration. The summary presented in Table 5.1 serves not only to encapsulate the findings of this study but also to encourage further inquiry and extension by interested readers.

The novel theorems provide crucial insights into the convergence conditions of the infinite series derived from these sequences. Moreover, by drawing upon the work of esteemed mathematicians such as Henry Gould, Rudi Lidl, Harald Niederreiter, and Morgan Ward, alongside the referenced work of Melham and Shannon, this chapter bridges contemporary research with established mathematical foundations, enriching the discourse on infinite series and their underlying properties.

CHAPTER 6

ON ALGORITHMS FOR REPRESENTING POSITIVE ODD INTEGERS AS THE SUM OF ARITHMETIC PROGRESSIONS

6.1 Background

Since 1844, there has been an interest in representing numbers as the sum of a sequence of consecutive integers. Initially, Sir Charles Wheatstone [94] represented certain powers of an integer as sums of arithmetic progressions. Then, Sylvester and Franklin [85] published a result to determine the number of ways a positive integer can be represented as the sum of a sequence of consecutive integers; this result has since been called Sylvester's Theorem, and there have been many attempts to extend this theorem to sums of different types of sequences, such as sums of certain arithmetic progressions (Munagi and de Vega [60], Munagi and Shonhiwa [61]) and sums of powers of arithmetic progressions (Shiue et al. [78], Shiue et al. [79]).

Recently, the manuscript duology of Ho et al. [29] and Ho et al. [30] extended Sylvester's Theorem to describe a procedure to compute the number of ways a positive integer can be represented as a sum of arithmetic progressions. They also extended Wheatstone's original work by studying certain relationships among the representations of different powers of an integer as sums of arithmetic progressions; this is done by using the method delineated in Junaidu et al. [40].

In this chapter, we present theorems and algorithms that enable us to represent positive odd integers m as arithmetic progressions of the form $m = a + (a+d) + \cdots + (a+(r-1)d)$. In Section 6.2.1 and 6.2.2, we explore the case where r is odd and even, respectively. Two corollaries follow that outline how many ways powers of primes can be written as arithmetic progressions under certain conditions. We provide computationally efficient algorithms corresponding to the theorems and corollaries mentioned in their respective sections.

6.2 Main Results

Let $m > 1$ be a positive odd integer represented as a sum of an arithmetic progression, i.e.,

$$m = a + (a + d) + \cdots + (a + (r - 1)d), \tag{6.1}$$

where $a, d \in \mathbb{N}$. In this section, we present algorithms for computing the number of ways that m can be represented as (6.1) when r is an odd integer ≥ 3 and r is an even integer > 2.

6.2.1 $r \geq 3$ odd

Theorem 6.2.1. *Let $m > 1$ be a positive odd integer, not a prime, and let*

$$m = a + (a + d) + \cdots + (a + (r - 1)d), \tag{6.2}$$

where $a, d \in \mathbb{N}$ and r is odd ≥ 3. Then,

(i). $r \mid m$, $1 \leq d \leq \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor$, and $a = \frac{m}{r} - \frac{r-1}{2}d$;

(ii). $3 \leq r \leq \left\lfloor \frac{-1+\sqrt{1+8m}}{2} \right\rfloor \leq \left\lfloor \sqrt{2m} \right\rfloor$;

(iii). There are

$$S = \sum_{r \mid m} \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor \tag{6.3}$$

number of ways to write m as (6.2).

Proof. (i). By (6.2), we have $m = r\left(a + \left(\frac{r-1}{2}\right)d\right)$. Then, $2m = r(2a + (r-1)d)$, or $\frac{2m}{r} = 2a + (r-1)d$. Since r is odd, we have $r \mid m$ and $\frac{m}{r} = a + \frac{r-1}{2}d$. Solving for a, then $a = \frac{m}{r} - \frac{r-1}{2}d$. Since $a \geq 1$, we have $\frac{m}{r} - \frac{r-1}{2}d \geq 1$, which implies $1 \leq d \leq \frac{2(m-r)}{r(r-1)}$.

(ii). From $\frac{2(m-r)}{r(r-1)} \geq 1$, we have $2(m-r) \geq r(r-1)$. Then $2m - 2r \geq r^2 - r$. Hence, $r^2 + r - 2m < 0$. Since $r \geq 3$, we have $3 \leq r \leq \left\lfloor \frac{-1+\sqrt{1+8m}}{2} \right\rfloor$. Since m is positive,

$4\sqrt{2m} \geq 0$. Then $4\sqrt{2m} + 8m + 1 = \left(2\sqrt{2m} + 1\right)^2 \geq 8m + 1$. Then $2\sqrt{2m} + 1 \geq \sqrt{8m+1}$. Simplifying this, we have $\left\lfloor \frac{-1 + \sqrt{1+8m}}{2} \right\rfloor \leq \left\lfloor \sqrt{2m} \right\rfloor$.

(iii). For each $r \mid m$, each d between 1 and $\left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor$ is a way to represent m as an arithmetic progression. To find total number of ways, the sum is taken.

□

Note that if $r = 1$, we have the arithmetic progression reduced to $m = a$. Hence, r is assumed to be odd and ≥ 3.

When $m = p$, where p is an odd prime number, we have the following corollary.

Corollary 6.2.2. *Let $m = p^k$, $p \geq 3$ a prime number and $k > 1$ an integer, and let*

$$p^k = a + (a + d) + \cdots + (a + (r-1)d), \tag{6.4}$$

where $a, d \in \mathbb{N}$, $r \geq 3$ odd. Then,

(i). $r = p^j$, $1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor$;

(ii). $1 \leq d \leq 2 \left\lfloor \frac{p^{k-j} - 1}{p^j - 1} \right\rfloor$ and $a = p^{k-j} - \frac{1}{2}\left(p^j - 1\right) d$;

(iii). There are

$$S = \sum_{j=1}^{\left\lfloor \frac{k}{2} \right\rfloor} 2 \left\lfloor \frac{\left(p^{k-j} - 1\right)}{p^j - 1} \right\rfloor \tag{6.5}$$

number of ways to write p^k as (6.4).

Proof. Since $m = p^k$, where p is a prime, then $r \mid m$ means that $r \mid p^k$. Hence, we have $r = p^j$, $1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor$. Substitute $m = p^k$ into the results in Theorem 6.2.1, we obtain the results for this corollary. □

Another way of listing the ways that a positive odd number can be represented in the form of (6.2) is as follows:

Remark 6.2.1. *Let $m > 1$ be a positive odd integer, not a prime, and represented by 6.2. Let $M = \frac{m}{r}$, where r is odd ≥ 3. Then there are*

$$S = \sum_{r|m} d \tag{6.6}$$

number of ways to write m as (6.2), where $1 \leq d \leq 2 \left\lfloor \frac{M-1}{r-1} \right\rfloor$.

Proof. Since $m = a + (a+d) + \cdots + (a + (r-1)d)$, where r is odd ≥ 3, we may set up

$$m = \left(M - \frac{r-1}{2}d\right) + \cdots + M + (M+d) + \cdots + \left(M + \frac{r-1}{2}d\right).$$

Since $M - \frac{r-1}{2}d \geq 1$, we have $M - 1 \geq \frac{r-1}{2}d$. Since $d \geq 1$, we have

$$1 \leq d \leq 2 \left\lfloor \frac{M-1}{r-1} \right\rfloor.$$

In addition, by (i) of Theorem 6.2.1, we have $a = M - \frac{r-1}{2}d$. This gives each a.

Thus, we have

$$S = \sum_{r|m} d.$$

□

6.2.2 $r > 2$ even

Throughout this section, we consider even r. When $r = 2$, m is represented by the sum of two positive integers, which is trivial. When $r > 2$, we have the following result.

Theorem 6.2.3. *Let $m > 1$ be a positive odd integer expressed as (6.2), where $a, d \in \mathbb{N}$ and $r = 2t$, with $t > 1$. Then,*

(i). t and d are odd;

(ii). $m \geq 21$;

(iii). $t \mid m$ *and* $1 < t \leq \left\lfloor \frac{-1+\sqrt{1+4m}}{4} \right\rfloor \leq \lfloor \sqrt{m} \rfloor$*;*

(iv). $1 \leq d \leq \left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$ *and* $a = \frac{1}{2}\left(\frac{m}{t} - (2t-1)d\right)$*;*

(v). There are

$$S = \sum_{t|m,\ g\in\mathbb{E}} \frac{g}{2} + \sum_{t|m,\ g\in\mathbb{O}} \frac{g+1}{2}, \tag{6.7}$$

where $g = \left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$ *ways to write* m *as the form of* (6.2)*, where* g *is either* $\mathbb{E}$ *(even) or* $\mathbb{O}$ *(odd).*

Proof. (i). From (6.2), we have $m = r\left(a + \left(\frac{r-1}{2}d\right)\right)$. Next, $\frac{2m}{r} = 2a + (r-1)d$, we have $r \mid 2m$. Let $r = 2t$ with $t > 1$, then $\frac{m}{t} = 2a + (2t-1)d$. If m is odd, then t is odd. Note that $2a$ is even and $\frac{m}{t}$ is odd, so d is odd.

(ii). We consider the smallest numbers $a = 1,\ d = 1$, and $r = 6$. Thus, $m \geq 6(1 + \frac{5}{2}) = 21$.

(iii). From $\frac{m}{t} = 2a + (2t-1)d$, we have $\frac{m-2at}{t(2t-1)} = d$. Since $a \geq 1$, then $\frac{m-2at}{t(2t-1)} \leq \frac{m-2t}{t(2t-1)}$. Hence $1 \leq d \leq \left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$. Solving for a, we have $a = \frac{1}{2}\left(\frac{m}{t} - (2t-1)d\right)$.

(iv). From $\frac{m-2t}{t(2t-1)} \geq 1$, we have $t(2t-1) \leq m - 2t$. Then $2t^2 + t - m \leq 0$. Hence, $t \leq \left\lfloor \frac{-1+\sqrt{1+4m}}{4} \right\rfloor$. To further lower the upper bound, observe that since $0 \leq 8\sqrt{m}$, $0 \leq 8\sqrt{m} + 12m$. Then $4m + 1 \leq 8\sqrt{m} + 16m + 1 = \left(4\sqrt{m}+1\right)^2$. Simplifying this, we have $\left\lfloor \frac{-1+\sqrt{1+4m}}{4} \right\rfloor \leq \lfloor \sqrt{m} \rfloor$.

(v). Since d is odd, we need to count the number of odd numbers between 1 and $\left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$, inclusive. Let $g = \left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$. If g is even, then there are $\frac{g}{2}$ odd numbers. If g is odd, then there are $\frac{g+1}{2}$ odd numbers.

□

Corollary 6.2.4. *Let $m = p^k$, $p \geq 3$ a prime number and $k \in \mathbb{N}$, and let*

$$p^k = a + (a + d) + \cdots + (a + (r - 1)d), \tag{6.8}$$

where $a, d \in \mathbb{N}$, d odd, and $r > 2$ even. Then,

(i). $r = 2p^j$, $1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor$;

(ii). $1 \leq d \leq \left\lfloor \frac{p^{k-j} - 2}{2p^j - 1} \right\rfloor$ and $a = \frac{1}{2}\left(p^{k-j} - \left(2p^j - 1\right) d\right)$;

(iii). There are

$$S = \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \left\lfloor \frac{1}{2}\left(\left\lfloor \frac{p^{k-j} - 2}{2p^j - 1} \right\rfloor + 1\right) \right\rfloor \tag{6.9}$$

number of ways to write p^k as (6.4).

Proof. Since p is prime and $m = p^k$, then $r = 2t$ and $t \mid m$ gives $r = 2t = 2p^j$, $1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor$.

Next, since $r = 2p^j = 2t$ and $m = p^k$, we have $\left\lfloor \frac{m - 2t}{t(2t - 1)} \right\rfloor = \left\lfloor \frac{p^k - 2p^j}{p^j(2p^j - 1)} \right\rfloor = \left\lfloor \frac{p^{k-j} - 2}{2p^j - 1} \right\rfloor$.

Lastly, since d is odd, we have the number of odd numbers in the interval $1 \leq d \leq \left\lfloor \frac{p^{k-j} - 2}{2p^j - 1} \right\rfloor$ equal to $\left\lfloor \frac{1}{2}\left(\left\lfloor \frac{p^{k-j} - 2}{2p^j - 1} \right\rfloor + 1\right) \right\rfloor$. □

6.3 Algorithms

In this section, we present algorithms corresponding to our main results.

6.3.1 $r \geq 3$ odd

By using Theorem 6.2.1, we have the following Algorithm 17.

Algorithm 17 Finding the number of ways S to write m as a sum of arithmetic progression when r is odd and $3 \leq r \leq \left\lfloor \sqrt{2m} \right\rfloor$

Input: Positive odd integer m (not prime)

Output: S

Define integers $q = \left\lfloor \sqrt{2m} \right\rfloor$, $r = 3$, $S = 0$

while $r \leq q$ **do**

  if $m \equiv 0 \pmod{r}$ **then**

    $g = \left\lfloor \dfrac{2(m-r)}{r(r-1)} \right\rfloor$

    $S = S + g$

  end if

  $r = r + 2$

end while

There are S number of ways to write the given m as the form (6.2) when $r \geq 3$ is odd.

The overall time complexity of this algorithm is determined by the number of iterations of the while loop. In the worst case, the while loop iterates until r exceeds q. Therefore, the time complexity is $O\left(\frac{q}{2}\right) = O(q)$.

The next algorithm computes the number of ways an integer $m = p^k$ can be represented as a sum of arithmetic progressions based on Corollary 6.2.2.

Algorithm 18 Finding the number of ways S to write $m = p^k$ as a sum of arithmetic progression when $r \geq 3$ is odd

Input: Positive odd integer $m = p^k$, p prime, and $k > 1$ integer

Output: S

Define integers $q = \left\lfloor \dfrac{k}{2} \right\rfloor$, $j = 1$, $S = 0$

while $j \leq q$ **do**

  $g = 2 \left\lfloor \dfrac{p^{k-j} - 1}{p^j - 1} \right\rfloor$

  $S = S + g$

  $j = j + 1$

end while

There are S number of ways to write the given m as the form (6.2) when $r \geq 3$ is odd and $m = p^k$, where p is a prime and $k > 1$ integer.

The next algorithm lists all possible ways to write m as sums of arithmetic progressions for a given m and r from algorithm 17.

Algorithm 19 Listing the possible ways to write m as a sum of arithmetic progressions

Input: Positive odd integer m not prime.

Output: Prints all S ways

Define integers $q = \left\lfloor \sqrt{2m} \right\rfloor$, $r = 3$
while $r \leq q$ **do**
  $g = \left\lfloor \dfrac{2(m-r)}{r(r-1)} \right\rfloor$
  if $m \equiv 0 \pmod{r}$ and $g \neq 0$ **then**
    for $d = 1, \ldots, g$ **do**
      $a = \dfrac{m}{r} - \dfrac{(r-1)}{2} d$
      Print '$m = a + (a+d) + \cdots + (a + (r-1)d)$'
    end for
  end if
  $r = r + 2$
end while

Similarly, this algorithm also has the time complexity $O(q \cdot g) = O(q)$. Although there is a for loop inside the while loop, since the operations take $O(1)$ and g is a constant factor, we can factor it out of the big-O notation.

6.3.2 $r > 2$ even

Algorithm 20 Finding the number of ways to write m as a sum of arithmetic progression when $r = 2t$, with $t > 1$ odd

Input: Positive odd integer m (not prime)

Output: S

```
Define integers q = ⌊√m⌋, t = 3, S = 0
while t ≤ q do
    if m ≡ 0 (mod t) then
        g = ⌊(m − 2t) / (t(2t − 1))⌋
        if g (mod 2) ≡ 0 then
            S_1 = g/2
        else
            S_2 = (g + 1)/2
        end if
        S = S_1 + S_2
    end if
    t = t + 2
end while
There are S ways to write the given m as the form (6.2) when r = 2t, with t > 1 odd.
```

Similarly, this algorithm should achieve the time complexity $O(q)$.

The next algorithm is similar to Algorithm 18, with $r > 2$ is even.

Algorithm 21 Finding the number of ways S to write $m = p^k$ as a sum of arithmetic progression when $r > 2$ is even

Input: Positive odd integer $m = p^k$, p prime, and $k > 1$ integer

Output: S

Define integers $q = \left\lfloor \frac{k}{2} \right\rfloor$, $j = 1$, $S = 0$
while $j \leq q$ **do**
  $g = \left\lfloor \frac{1}{2}\left(\frac{p^{k-j}-2}{2p^j-1}+1\right)\right\rfloor$
  $S = S + g$
  $j = j + 1$
end while
There are S number of ways to write the given m as the form (6.2) when $r > 2$ is even and $m = p^k$, where p is a prime, and $k > 1$ integer.

The next algorithm lists all sums of arithmetic progressions for a given m and $r = 2t$, with $t > 1$ odd, from algorithm 20.

Algorithm 22 Listing the possible ways to write m, m odd not prime, as a sum of arithmetic progressions

Input: Positive odd integer m not prime.

Output: Prints all S ways

Initialize integers $q = \lfloor \sqrt{m} \rfloor$, $t = 3$
while $t \leq q$ **do**
  $g = \left\lfloor \frac{m-2t}{t(2t-1)} \right\rfloor$
  if $m \equiv 0 \pmod{t}$ and $g \neq 0$ **then**
    for $d = 1, \ldots, g$ **do**
      $a = \frac{1}{2}\left(\frac{m}{t} - (2t-1)d\right)$
      Print '$m = a + (a + d) + \cdots + (a + (r - 1)d)$'
    end for
  end if
  $t = t + 2$
end while

Similarly, this algorithm also has the time complexity $O(q \cdot g) = O(q)$.

6.4 Examples

Example 6.4.1. *Let $m = 65$. How many ways can we write m as a sum of a sum of arithmetic progressions?*

Case I. $r \geq 3$ odd. Use algorithm 17.

First, $q = \left\lfloor \sqrt{2m} \right\rfloor = \left\lfloor \sqrt{130} \right\rfloor = 11$ and $3 \leq r \leq 11$. Since $m = 65$, then $r = 5$. Computing the quantity $\sum_{r|m} \left\lfloor \frac{2(m-r)}{r(r-1)} \right\rfloor$ for $r = 5$, we have $S = 6$. Thus, there are 6 ways to write $m = 65$ as a sum of arithmetic progressions when $r \geq 3$ is odd.

To list the possible ways, we use algorithm 19. Now, for $r = 5$, $g = \left\lfloor \frac{2(65-5)}{5(5-1)} \right\rfloor = 6$. We iterate over each d from 1 to 6. Then

$$d = 1 \text{ gives } a = 11$$

$$d = 2 \text{ gives } a = 9$$

$$d = 3 \text{ gives } a = 7$$

$$d = 4 \text{ gives } a = 5$$

$$d = 5 \text{ gives } a = 3$$

$$d = 6 \text{ gives } a = 1$$

Then we have the following list:

$$65 = 11 + 12 + \cdots + 15$$

$$65 = 9 + 11 + \cdots + 17$$

$$65 = 7 + 10 + \cdots + 19$$

$$65 = 5 + 9 + \cdots + 21$$

$$65 = 3 + 8 + \cdots + 23$$

$$65 = 1 + 7 + \cdots + 25$$

Case II. $r = 2t$ with $t \in \mathbb{N}$ odd. Use algorithm 20.

First, $q = \left\lfloor \sqrt{65} \right\rfloor = 8$. Then $3 \leq t \leq 8$. The only t that divides $m = 65$ is $t = 5$. Now,

$g = 1$, there is only one way to write $m = 65$ as a sum of arithmetic progressions when $t = 5$. The parameters are $a = 2$ and $d = 1$, which gives

$$65 = 2 + 3 + \cdots + 11.$$

In total, there are 7 ways.

Example 6.4.2. *Let $m = 1125$. How many ways can we write m as a sum of arithmetic progressions?*

Case I. $r \geq 3$ odd. We use algorithm 17.

First, $q = \left\lfloor \sqrt{2m} \right\rfloor = \left\lfloor \sqrt{2250} \right\rfloor = 47$. Then, $3 \leq r \leq 47$. Since $m = 1125$, $r = 3, 5, 9, 15, 25, 45$. Now, let $g(r) = \left\lfloor \dfrac{2(m-r)}{r(r-1)} \right\rfloor$. Then,

r	$g(r)$
3	374
5	112
9	31
15	10
25	3
45	1
S	531

There are 531 ways to write $m = 1125$ as a sum of arithmetic progressions when $r \geq 3$ is odd. To list the possible ways, we use algorithm 19.

For $r = 3$, there is a total of 374 ways $(374 + 1 - 1 = 374)$:

$$1125 = 374 + 375 + 376$$

$$1125 = 373 + 375 + 377$$

$$\vdots$$

$$1125 = 2 + 375 + 748$$

$$1125 = 1 + 375 + 749$$

For $r = 5$, *there is a total of* 112 *ways* $\left(\frac{223-1}{2} + 1 = 112\right)$*:*

$$1125 = 223 + 224 + \cdots + 227$$

$$1125 = 221 + 223 + \cdots + 229$$

$$\vdots$$

$$1125 = 3 + 114 + \cdots + 447$$

$$1125 = 1 + 113 + \cdots + 449$$

For $r = 9$, *there is a total of* 31 *ways* $\left(\frac{121-1}{4} + 1 = 31\right)$*:*

$$1125 = 121 + 122 + \cdots + 129$$

$$1125 = 117 + 119 + \cdots + 133$$

$$\vdots$$

$$1125 = 5 + 35 + \cdots + 245$$

$$1125 = 1 + 32 + \cdots + 249$$

For $r = 15$, *there is a total of* 10 *ways* $\left(\frac{68-5}{7} + 1 = 10\right)$*:*

$$1125 = 68 + 69 + \cdots + 82$$

$$1125 = 61 + 63 + \cdots + 89$$

$$\vdots$$

$$1125 = 12 + 21 + \cdots + 138$$

$$1125 = 5 + 15 + \cdots + 145$$

For $r = 25$, *there is a total of* 3 *ways:*

$$1125 = 33 + 34 + \cdots + 57$$

$$1125 = 21 + 23 + \cdots + 69$$

$$1125 = 9 + 12 + \cdots + 81$$

For $r = 45$, there is only 1 way:

$$1125 = 3 + 4 + \cdots + 47$$

Case II. $r = 2t$, with $t \in \mathbb{N}$ odd. We use algorithm 20.

First, $q = \lfloor \sqrt{m} \rfloor = \left\lfloor \sqrt{1125} \right\rfloor = 33$. Then, $3 \leq t \leq 33$. Since $m = 1125$, the possible values of t are $3, 5, 9, 15, 25$. Then,

t	$g(t)$
3	74
5	24
9	7
15	2
25	1
S	$54 = \frac{74}{2} + \frac{24}{2} + \frac{7+1}{2} + \frac{2}{2} + \frac{1+1}{2}$

There are 54 ways to write $m = 1125$ as a sum of arithmetic progressions when $r = 2t$, with $t \in \mathbb{N}$ odd.

In total, there are 585 ways.

Example 6.4.3. *Let $m = 6125$. How many ways can we write m as a sum of arithmetic progressions?*

Case I. $r \geq 3$ odd. We use algorithm 17.

First, $q = \left\lfloor \sqrt{2m} \right\rfloor = \left\lfloor \sqrt{12250} \right\rfloor = 110$. Then, $3 \leq r \leq 110$. Since $m = 6125$, $r = 5, 7, 25, 35, 49$.

Now, let $g(r) = \left\lfloor \dfrac{2(m-r)}{r(r-1)} \right\rfloor$. *Then,*

r	$g(r)$
5	612
7	291
25	20
35	10
49	5
S	938

There are 938 *ways to write* $m = 6125$ *as a sum of arithmetic progressions when* $r \geq 3$ *is odd. To list the possible ways, we use algorithm 19.*

For $r = 5$*, there is a total of* 612 *ways* $(1223 - 1 + 1 = 1223)$*:*

$$6125 = 1223 + 1224 + \cdots + 1227$$

$$6125 = 1221 + 1223 + \cdots + 1229$$

$$\vdots$$

$$6125 = 3 + 614 + \cdots + 2447$$

$$6125 = 1 + 613 + \cdots + 2449$$

For $r = 7$*, there is a total of* 291 *ways* $\left(\dfrac{872 - 2}{3} + 1 = 291 \right)$*:*

$$6125 = 872 + 873 + \cdots + 878$$

$$6125 = 869 + 871 + \cdots + 881$$

$$\vdots$$

$$6125 = 5 + 295 + \cdots + 1745$$

$$6125 = 2 + 293 + \cdots + 1748$$

For $r = 25$*, there is a total of* 20 *ways* $\left(\dfrac{223-5}{12}+1=20\right)$*:*

$$6125 = 233 + 234 + \cdots + 257$$

$$6125 = 221 + 223 + \cdots + 269$$

$$\vdots$$

$$6125 = 17 + 36 + \cdots + 473$$

$$6125 = 5 + 25 + \cdots + 485$$

For $r = 35$*, there is a total of* 10 *ways* $\left(\dfrac{158-5}{17}+1=10\right)$*:*

$$6125 = 158 + 159 + \cdots + 192$$

$$6125 = 141 + 143 + \cdots + 209$$

$$\vdots$$

$$6125 = 22 + 31 + \cdots + 328$$

$$6125 = 5 + 15 + \cdots + 345$$

For $r = 45$*, there is a total of* 5 *ways:*

$$6125 = 101 + 102 + \cdots + 149$$

$$6125 = 77 + 79 + \cdots + 173$$

$$6125 = 53 + 56 + \cdots + 197$$

$$6125 = 29 + 33 + \cdots + 221$$

$$6125 = 5 + 10 + \cdots + 245$$

Case II. $r = 2t$*, with* $t \in \mathbb{N}$ *odd. We use algorithm 20.*

First, $q = \lfloor \sqrt{m} \rfloor = \left\lfloor \sqrt{6125} \right\rfloor = 78$*. Then,* $3 \leq t \leq 78$*. Since* $m = 6125$*,* $t = 5, 7, 25, 35, 49$*. Then,*

t	$g(t)$
5	135
7	67
25	4
35	2
49	1
S	$106 = \dfrac{135+1}{2} + \dfrac{67+1}{2} + \dfrac{4}{2} + \dfrac{2}{2} + \dfrac{1+1}{2}$

There are 106 *ways to write* $m = 6125$ *as a sum of arithmetic progressions when* $r = 2t$*, with* $t \in \mathbb{N}$ *odd.*

In total, there are 1044 *ways.*

Example 6.4.4. *Let* $m = 7^5$ *(Ho et al. [29]). How many ways can we write* m *as a sum of arithmetic progressions?*

Case I. $r \geq 3$ *odd. Using Algorithm 18, we have* $p = 7$*,* $k = 5$*,* $r = 7^j$*,* $\left\lfloor \dfrac{k}{2} \right\rfloor = 2$*, and* $j = 1, 2$*. Let* $g(j) = \left\lfloor \dfrac{2(7^{5-j} - 1)}{7^j - 1} \right\rfloor$*. Then,*

j	$g(j)$
1	800
2	14
S	814

There are 814 *number of ways of writing* 7^5 *as Eq.* (6.4) *when* $r \geq 3$ *is odd. To list the possible ways, we use algorithm 19.*

For $j = 1$*, there is a total of* 800 *ways* $\left(\dfrac{2398 - 1}{3} + 1 = 800\right)$*:*

$$16807 = 2398 + 2399 + \cdots + 2404$$

$$16807 = 2395 + 2397 + \cdots + 2407$$

$$\vdots$$

$$16807 = 4 + 803 + \cdots + 4798$$

$$16807 = 1 + 801 + \cdots + 4801$$

For $j = 2$, there is a total of 14 ways $\left(\frac{319 - 7}{24} = 14\right)$*:*

$$16807 = 319 + 320 + \cdots + 367$$

$$16807 = 295 + 297 + \cdots + 391$$

$$\vdots$$

$$16807 = 31 + 44 + \cdots + 655$$

$$16807 = 7 + 21 + \cdots + 679$$

Case II. $r = 2t$, $t \in \mathbb{N}$ odd. Using Algorithm 21, we have $q = \left\lfloor \frac{5}{2} \right\rfloor = 2$*. Then $j = 1, 2$. Let* $g(j) = \left\lfloor \frac{1}{2} \left(\frac{p^{k-j} - 2}{2p^j - 1} + 1 \right) \right\rfloor$.

Then,

j	$g(j)$
1	92
2	2
S	94

There are 94 ways to write 7^5 as Eq. (6.4) when $r > 2$ is even.

In total, there are 908 ways.

Example 6.4.5. *Let $m = 3^{10}$. How many ways can we write m as a sum of arithmetic progressions?*

Case I. $r \geq 3$ odd.

Using Algorithm 18, we have $r = 3^j$, $\left\lfloor \frac{10}{2} \right\rfloor = 5$*, and $j = 1, 2, \cdots, 5$. Let* $g(j) = \left\lfloor \frac{2(7^{5-j} - 1}{7^j - 1} \right\rfloor$.

Then,

j	$g(j)$
1	19682
2	1640
3	168
4	18
5	2
S	21510

Thus, there are 21510 *ways of writing* 3^{10} *as Eq.* (6.4) *when* $r \geq 3$ *is odd.*

Case II. $r > 2$ *even.*

Using Algorithm 21, we have $r = 2 \cdot 3^j$, $j = 1, 2, \cdots, 5$. *Let* $g(j) = \left\lfloor \frac{1}{2} \left(\frac{p^{k-j} - 2}{2p^j - 1} + 1 \right) \right\rfloor$.

Then,

j	$g(j)$
1	1968
2	193
3	21
4	2
5	0
S	2184

There are 2184 *ways of writing* 3^{10} *as Eq.* (6.4) *when* $r > 2$ *is even.*

In total, there are 23694 *ways.*

6.5 Conclusion

We presented novel theorems and algorithms concerning the representation of a positive odd integer m as arithmetic progressions. The presented results greatly expand upon the work of the manuscript duology of Ho et al. [29] and Ho et al. [30], co-authored by Professor Chungwu Ho to whom this manuscript is dedicated. Historically, there has been a great interest in representing integers and powers of integers as sums of arithmetic progressions. Notably, Sylvester's Theorem has received much recent attention in the field of number theory. We extended Sylvester's Theorem to include results for all positive odd m.

We solved the problem of counting the total number of ways one can represent m as a sum of arithmetic progressions. In doing so, we considered the two distinct cases of odd r and even r. We found that partitioning the solution in such a way results in convenient mathematical results and highly efficient computational algorithms; the mathematical convenience motivates a further exploration of sums of arithmetic progressions, and the efficient algorithms encourages the adoption of the presented results. Representing positive even integers as the sum of arithmetic progressions will be a focus of future work.

BIBLIOGRAPHY

[1] M. Aigner. *Discrete Mathematics*. American Mathematical Soc., 2010.

[2] Y. Alp. Hybrid hyper-Fibonacci and hyper-Lucas numbers. *Notes on Number Theory and Discrete Mathematics*, 29(1):154–170, 2023.

[3] M. A. Asbullah and M. R. K. Ariffin. Design of rabin-like cryptosystem without decryption failure. *Malaysian Journal of Mathematical Sciences*, 10:1–18, 2016.

[4] E. Bach and J. Shallit. *Algorithmic Number Theory: Efficient Algorithms*, volume 1. MIT press, 1996.

[5] M. Bahşi and S. Solak. A symmetric algorithm for golden ratio in hyper-Horadam numbers. *Chinese Journal of Mathematics*, 2016.

[6] P. S. Barreto and J. F. Voloch. Efficient computation of roots in finite fields. *Designs, Codes and cryptography*, 39(2):275–280, 2006.

[7] B. C. Berndt and S. Bhargava. Ramanujan—for lowbrows. *The American mathematical monthly*, 100(7):644–656, 1993.

[8] M. Bicknell-Johnson and G. E. Bergum. The generalized Fibonacci numbers $\{C_n\}$, $C_n = C_{n-1} + C_{n-2} + k$. In *Applications of Fibonacci Numbers: Proceedings of The Second International Conference on Fibonacci Numbers and Their Applications*, pages 193–205. Springer, 1988.

[9] F. Bounebirat, D. Laissaoui, and M. Rahmani. Several explicit formulae of sums and hyper-sums of powers of integers. *Online Journal of Analytic Combinatorics*, 13, 2018.

[10] L. Carlitz. Eulerian numbers and polynomials. *Mathematics Magazine*, 32(5):247–260, 1959. ISSN 0025570X, 19300980. URL http://www.jstor.org/stable/3029225.

[11] P. M. Catarino and A. Borges. On Leonardo numbers. *Acta Mathematica Universitatis Comenianae*, 89(1):75–86, 2019.

[12] C. A. Charalambides. *Enumerative Combinatorics*. Chapman & Hall / CRC Press, 2002.

[13] W. Y. Chen, A. M. Fu, and I. F. Zhang. Faulhaber's theorem on power sums. *Discrete Mathematics*, 309(10):2974–2981, 2009.

[14] W. Y. C. Chen. Cubic equations through the looking glass of Sylvester. *To Appear in College Mathematics Journal*, 2021.

[15] Y.-S. Chen and Y.-W. Lin. C-mac: an energy-efficient mac scheme using chinese-remainder-theorem for wireless sensor networks. In *2007 IEEE International Conference on Communications*, pages 3576–3581. IEEE, 2007.

[16] L. Childs. *A Concrete Introduction to Higher Algebra*. Springer, 1995.

[17] H. V. Chu, N. Irmak, S. J. Miller, L. Szalay, and S. X. Zhang. Schreier multisets and the s-step Fibonacci sequences. *In press*, 2023.

[18] M. Cipolla. Un metodo per la risolutione della congruenza di secondo grado. *Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche, Napoli*, 9:154–163, 1903.

[19] L. Comtet. *Advanced Combinatorics: The art of finite and infinite expansions*. Springer Science & Business Media, 1974.

[20] A. Dil and I. Mező. A symmetric algorithm for hyperharmonic and Fibonacci numbers. *Applied Mathematics and Computation*, 206(2):942–951, 2008.

[21] P. Dingyi, S. Arto, and D. Cunsheng. *Chinese remainder theorem: applications in computing, coding, cryptography*. World Scientific, 1996.

[22] G. Dresden, P. Panthi, A. Shrestha, and J. Zhang. Cubic polynomials, linear shifts, and Ramanujan simple cubics. *Mathematics Magazine*, 92(5):374–381, 2019.

[23] A. Frank. Uci machine learning repository. *http://archive.ics.uci.edu/ml*, 2010.

[24] L. Gilbert and J. Gilbert. *Elements of modern algebra.* Cengage Learning, 8 edition, (2014).

[25] H. W. Gould. The Girard-Waring power sum formulas for symmetric functions, and Fibonacci sequences. *Fibonacci Quarterly*, 37:135–140, 1999.

[26] T. Granlund et al. *GNU MP 6.0 Multiple precision arithmetic library.* Samurai Media Limited, 2015.

[27] T.-X. He and P. J.-S. Shiue. On sequences of numbers and polynomials defined by linear recurrence relations of order 2. *International Journal of Mathematics and Mathematical Sciences*, 2009:21, 2009. Article ID 709386.

[28] T.-X. He, P. J.-S. Shiue, and H.-C. Liao. Matrix representation of recursive sequences of order 3 and its applications. *Journal of Mathematical Research with Applications*, 38(3):221–235, 2018.

[29] C. Ho, T.-X. He, and P. J.-S. Shiue. Representations of positive integers as sums of arithmetic progressions, I. *Notes on Number Theory and Discrete Mathematics*, 29(2):241–259, 2023.

[30] C. Ho, T.-X. He, and P. J.-S. Shiue. Representations of positive integers as sums of arithmetic progressions, II. *Notes on Number Theory and Discrete Mathematics*, 29(2):260–275, 2023.

[31] A. Horadam. Generalization of two theorems of K. Subba Rao. *Bulletin of Calcutta Mathematical Society*, 58(1):23–29, 1966.

[32] A. Horadam. Special properties of the sequence $w_n(a, b; p, q)$. *Fibonacci Quarterly*, 5(5):424–434, 1967.

[33] A. F. Horadam. Basic properties of a certain generalized sequence of numbers. *The Fibonacci Quarterly*, 3(3):161–176, 1965.

[34] L. C. Hsu and P. J. Shiue. On certain summation problems and generalizations of Eulerian polynomials and numbers. *Discrete Mathematics*, 204(1-3):237–247, 1999.

[35] S. C. Huang, P. J. Shiue, A. G. Shannon, and J. Reyes. Algorithms for computing ramanujan-type identities from cubic equations. In *2023 Joint Mathematics Meetings (JMM 2023)*. AMS, 2023.

[36] S. C. Huang, A. G. Shannon, and J.-S. S. Peter. Notes on generalized and extended leonardo numbers. In *2024 Joint Mathematics Meetings (JMM 2024)*. AMS, 2024.

[37] R. H. Hudson. Convergence of tribonacci decimal expansions. *Fibonacci Quarterly*, 25:163–70, 1987.

[38] D. Jarden. Recurring sequences. *Jerusalem: Riveon Lematematika*, pages 95–102, 1996.

[39] B. Javidi, A. Carnicer, M. Yamaguchi, T. Nomura, E. Pérez-Cabré, M. S. Millán, N. K. Nishchal, R. Torroba, J. F. Barrera, W. He, et al. Roadmap on optical security. *Journal of Optics*, 18(8):083001, 2016.

[40] S. Junaidu, A. Laradji, and A. Umar. Powers of integers as sums of consecutive odd numbers. *The Mathematical Gazette*, 94(529):117–119, 2010.

[41] J. Katz, A. J. Menezes, P. C. Van Oorschot, and S. A. Vanstone. *Handbook of applied cryptography.* CRC press, 1996.

[42] D. E. Knuth. Johann Faulhaber and sums of powers. *Mathematics of Computation*, 61(203): 277–294, 1993.

[43] T. Komatsu and L. Szalay. A new formula for hyper-Fibonacci numbers, and the number of occurrences. *Turkish Journal of Mathematics*, 42(3):993–1004, 2018.

[44] T. Koshy. *Fibonacci and Lucas Numbers with Applications, Volume 2.* John Wiley & Sons, 2019.

[45] K. Kuhapatanakul and J. Chobsorn. On the generalized Leonardo numbers. *Integers*, 22(A48), 2022.

[46] R. Kumanduri and C. Romero. *Number theory with computer applications.* Pearson College Division, 1998.

[47] D. Laissaoui and M. Rahmani. An explicit formula for sums of powers of integers in terms of Stirling numbers. *Journal of Integer Sequences*, 20(4), 2017. Aricle 17.4.8.

[48] D. Lehmer. Generalized Eulerian numbers. *Journal of Combinatorial Theory, Series A*, 32(2): 195–215, 1982.

[49] X. Li, W. Wang, W. Zhang, and Y. Cao. Phase-detection-based range estimation with robust chinese remainder theorem. *IEEE Transactions on Vehicular Technology*, 65(12):10132–10137, 2016.

[50] H.-C. Liao, M. Saul, and P. J.-S. Shiue. Revisiting the general cubic: A simplification of Cardano's solution. *The Mathematical Gazette*, 2022.

[51] R. Lidl and H. Niederreiter. *Finite fields*. Number 20. Cambridge university press, 1997.

[52] D. Lind. On a class of non-linear binomial sums. *The Fibonacci Quarterly*, 3(4):292–298, 1965.

[53] S. Lindhurst. An analysis of shanks' algorithm for computing square roots in finite fields. In *Number theory (Ottawa, 1996), CRM Proc. Lecture Notes*, volume 19, pages 231–242, 1999.

[54] Y.-C. Liu and P. J.-S. Shiue. Convergent series of a second-order linear recursive sequence. *Math Media*, 44(3):31 – 41, 2022. (In Chinese).

[55] C. T. Long. The decimal expansion of 1/89 and related results. *Fibonacci Quarterly*, 19(1):53 – 56, 1981.

[56] J. Lopez and R. Dahab. An overview of elliptic curve cryptography. *Institute of Computing, Sate University of Campinas, Sao Paulo, Brazil, Tech. Rep*, 2000.

[57] Z. Mahad, M. Asbullah, and M. Ariffin. Efficient methods to overcome Rabin cryptosystem decryption failure. *Malaysian Journal of Mathematical Sciences*, 11:9–20, 2017.

[58] R. Melham and A. G. Shannon. Some infinite series summations using power-series evaluated at a matrix. *Fibonacci Quarterly*, 33(1):13–20, 1995.

[59] R. A. Mollin. *An introduction to cryptography*. Chapman and Hall/CRC, 2006.

[60] A. O. Munagi and F. J. de Vega. An extension of sylvester's theorem on arithmetic progressions. *Symmetry*, 15(6):1276, 2023.

[61] A. O. Munagi and T. Shonhiwa. On the partitions of a number into arithmetic progressions. *Journal of Integer Sequences*, 11(2):3, 2008.

[62] K. Nagasaka, J.-S. Shiue, and C.-W. Ho. A fast algorithm of the Chinese remainder theorem and its application to Fibonacci numbers. In *Applications of Fibonacci Numbers*, volume 4, pages 241–246. Springer, 1991.

[63] I. Niven, H. S. Zuckerman, and H. L. Montgomery. *An introduction to the theory of numbers.* John Wiley & Sons, 1991.

[64] R. L. Ollerton and A. G. Shannon. Some properties of generalized Pascal squares and triangles. *The Fibonacci Quarterly*, 36(2):98–109, 1998.

[65] G. Panda and P. K. Ray. Some links of balancing and cobalancing numbers with Pell and associated Pell numbers. *Bulletin of the Institute of Mathematics, Academia Sinica (New Series)*, 6(1):41–72, 2011.

[66] C. Pita-Ruiz. On a generalization of Eulerian numbers. *Notes on Number Theory and Discrete Mathematics*, 24(1):16–42, 2018.

[67] M. O. Rabin. Digitalized signatures and public-key functions as intractable as factorization. Technical report, Massachusetts Inst of Tech Cambridge Lab for Computer Science, 1979.

[68] S. Ramanujan. *Notebooks of Srinivasa Ramanujan (2 Volumes)*. Tata Institute of Fundamental Research, Bombay, (1957).

[69] K. H. Rosen. *Elementary number theory.* Pearson Education, 2011.

[70] D. Shanks. Five number-theoretic algorithms. In *Proceedings of the Second Manitoba Conference on Numerical Mathematics (Winnipeg), 1973*, pages 51–70, 1973.

[71] A. Shannon. Tribonacci numbers and Pascal's pyramid. *Fibonacci Quarterly*, 15(3):268–275, 1977.

[72] A. Shannon, P. G. Anderson, and A. Horadam. Properties of cordonnier, perrin and van der laan numbers. *International Journal of Mathematical Education in Science and Technology*, 37(7):825–831, 2006.

[73] A. G. Shannon. A note on generalized Leonardo numbers. *Notes on Number Theory and Discrete Mathematics*, 25(3):97–101, 2019.

[74] A. G. Shannon and Ö. Deveci. A note on generalized and extended Leonardo sequences. *Notes on Number Theory and Discrete Mathematics*, 28(1):109–114, 2022.

[75] A. G. Shannon, P. J.-S. Shiue, and S. C. Huang. Notes on generalized and extended Leonardo numbers. *Notes on Number Theory and Discrete Mathematics*, 29(4):751 –773, 2023.

[76] M. Shattuck. Combinatorial proofs of identities for the generalized Leonardo numbers. *Notes on Number Theory and Discrete Mathematics*, 28(4):778–790, 2022.

[77] V. Shevelev. On Ramanujan cubic polynomials. *South East Asian Mathematics and Mathematical Sciences*, pages 113–122, 2007.

[78] P. J. Shiue, S. C. Huang, and E. Jameson. On algorithms for computing the sums of powers of arithmetic progressions. *Notes on Number Theory and Discrete Mathematics*, 26(4):113–121, 2020.

[79] P. J. Shiue, S. C. Huang, and J. E. Reyes. Algorithms for computing sums of powers of arithmetic progressions by using eulerian numbers. *Notes on Number Theory and Discrete Mathematics*, 27:140, 2021.

[80] P. J.-S. Shiue, S. C. Huang, and E. Jameson. On algorithms for computing the sums of powers of arithmetic progressions. *Notes on Number Theory and Discrete Mathematics*, 26(4):113–121, 2020.

[81] P. J.-S. Shiue, A. G. Shannon, S. C. Huang, and J. E. Reyes. Notes on efficient computation of Ramanujan cubic equations. *Notes on Number Theory and Discrete Mathematics*, 28(2): 350–370, 2022.

[82] P. J.-S. Shiue, A. G. Shannon, S. C. Huang, and J. E. Reyes. Notes on efficient computation of ramanujan cubic equations. *Notes on Number Theory and Discrete Mathematics*, 28(2): 350–375, 2022.

[83] P. J.-S. Shiue, A. G. Shannon, S. C. Huang, and J. E. Reyes. A generalized computation procedure for ramanujan-type identities and cubic shevelev sum. *Notes on Number Theory and Discrete Mathematics*, 29(1):98–129, 2023.

[84] N. J. Sloane et al. The on-line encyclopedia of integer sequences, 2008.

[85] J. J. Sylvester and F. Franklin. A constructive theory of partitions, arranged in three acts, an interact and an exodion. *American Journal of mathematics*, 5(1):251–330, 1882.

[86] T. Takagi. Fast RSA-type cryptosystems using n-adic expansion. In *Annual International Cryptology Conference*, pages 372–384. Springer, 1997.

[87] A. Tonelli. Bemerkung über die auflösung quadratischer congruenzen. *Nachrichten von der Königl. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen*, 1891:344–346, 1891.

[88] C. Wang, Q. Yin, and H. Chen. Robust Chinese remainder theorem ranging method based on dual-frequency measurements. *IEEE Transactions on Vehicular Technology*, 60(8):4094–4099, 2011.

[89] K. Wang. On cubic equations with zero sums of cubic roots of roots. *preprint*, 2019.

[90] K. Wang. On Ramanujan type identities and Cardano formula. *Notes on Number Theory and Discrete Mathematics*, 3(27):155—174, 2021.

[91] W. Wang and X.-G. Xia. A closed-form robust Chinese remainder theorem and its performance analysis. *IEEE Transactions on Signal Processing*, 58(11):5655–5666, 2010.

[92] W. Wang, X. Li, X.-G. Xia, and W. Wang. The largest dynamic range of a generalized Chinese remainder theorem for two integers. *IEEE Signal Processing Letters*, 22(2):254–258, 2014.

[93] M. Ward. Memoir on elliptic divisibility sequences. *American Journal of Mathematics*, 70(1): 31–74, 1948.

[94] C. Wheatstone. Beschreibung verschiedener neuen instrumente und methoden zur bestimmung der constanten einer volta'schen kette. *Annalen der Physik*, 138(8):499–543, 1844.

[95] R. Wituła. Full description of Ramanujan cubic polynomials. *Journal of Integer Sequences*, 13, 2010. Article 10.5.7.

[96] J. Worpitzky. Studien über die Bernoullischen und Eulerschen Zahlen. *Journal für die reine und angewandte Mathematik*, 94:203–232, 1883.

[97] T. Xiong, H.-P. Tsao, and J. I. Hall. General Eulerian numbers and Eulerian polynomials. *Journal of Mathematics*, 2013:1–9, 2013.

[98] A. Yatish, Fatima and N. K. Nishchal. Optical image encryption using triplet of functions. *Optical Engineering*, 57(3):033103, 2018.

[99] J. Zhan, P. J.-S. Shiue, S. C. Huang, and B. J. Lowe. Towards a novel generalized Chinese remainder algorithm for extended Rabin cryptosystem. *IEEE Access*, 8:26433–26444, 2020.

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3. Shannon, A. G., Shiue, P. J.-S., **Huang, S. C.**, "Notes on Generalized and Extended Leonardo Numbers", Notes on Number Theory and Discrete Mathematics, 29(4), 751–773, DOI: 10.7546/nntdm.2023.29.4.752-773

4. Shiue, P. J.-S., Shannon, A. G., **Huang, S. C.**, & Reyes, J. E. "A Generalized Computation Procedure for Ramanujan-type Identities and Cubic Shevelev Sum", Notes on Number Theory and Discrete Mathematics, 29(1), 98–129, DOI: 10.7546/nntdm.2023.29.1.98-129

5. Shiue, P. J.-S., Shannon, A. G., **Huang, S. C.**, & Reyes, J. E. (2022). "Notes on Efficient Computation of Ramanujan Cubic Equation." Notes on Number Theory and Discrete Mathematics, 28(2), 350-375, DOI: 10.7546/nntdm.2022.28.2.350-375.

6. Shiue, P. J.-S., **Huang, S. C.**, & Reyes, J. E., "Algorithms for Computing Sums of Powers of Arithmetic Progressions by Using Eulerian Numbers," Notes on Number Theory and Discrete Mathematics, 27(4), 140-148, 2021, DOI: 10.7546/nntdm.2021.
27.4.140-148.

7. Shiue, P. J.-S., **Huang, S. C.**, & Jameson, E., "On Algorithms for Computing the Sums of Powers of Arithmetic Progressions," Notes on Number Theory and Discrete Mathematics, 27(4), 113 - 121, 2020, DOI: 10.7546/nntdm.2020.26.4.113-121.

8. Zhan, J, Shiue, P J.-S., **Huang, S. C.** , and Lowe B. J., "Towards a Novel Generalized Chinese Remainder Algorithm for Extended Rabin Cryptosystem," in IEEE Access, vol. 8, pp. 26433-26444, 2020, DOI: 10.1109/ACCESS.2020.2967396.

Presentations:

1. **Huang, S.C.***, Shiue, Peter J.-S., and Shannon, Anthony G., "Notes on generalized and extended Leonardo sequences and series," Joint Mathematics Meetings, San Francisco, CA, 2024.

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4. **Huang, S.C.***, Shiue, Peter J.-S., Shannon, Anthony G., and Reyes, J.E., "Notes on Computing Ramanujan-Type Identities from Cubic Equations,", GPSA Research Forum, University of Nevada, Las Vegas, 2023.

5. **Huang, S.C.***, Shiue, Peter J.-S., Shannon, Anthony G., and Reyes, J.E., "Algorithms for Computing Ramanujan-Type Identities from Cubic Equations," Joint Mathematics Meetings, Boston, MA, 2023.

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9. P.J. Shiue, **Huang, S.C.**, and Reyes, J.E.*,"On Sums of Powers - Eulerian Numbers," Graduate & Professional Student Research Forum, 2^{nd} place, University of Nevada, Las Vegas, 2021.

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