GLOBAL STRUCTURE AND ASYMPTOTIC PROFILES OF THE ENDEMIC EQUILIBRIA OF A DIFFUSIVE EPIDEMIC MODEL WITH MASS-ACTION

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ABSTRACT

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Infectious diseases are a great challenge to the health and successful function of society. Therefore, it becomes crucial to develop methods and tools that would allow us to be able to control an infectious disease once it starts spreading within an environment. In this regard, mathematical research on epidemic models has provided important tools in the qualitative and quantitative analysis of the spread and control of infectious diseases. Each mathematical epidemic model incorporates important factors that could affect the spread of a disease, such as population movement and temporal or environmental heterogeneity.

This dissertation focuses on a susceptible-infected-susceptible (SIS) model in the form of a system of diffusive partial differential equations that takes into account a moving population within a spatially heterogeneous environment. Our goal is to assess the effectiveness of disease control strategies aimed at restricting population movement. To this end, we first consider basic fundamental questions such as existence, uniqueness, and global stability of solutions to the model. Next, we discuss how population movement may affect the disease dynamics by looking at the asymptotic profiles of endemic equilibrium (EE) solutions of the model. Consequently, we determine conditions leading to a multiplicity of EE solutions, which demonstrate that the disease can become difficult to control when movement is included in the model. In doing so, we discover various bifurcation curves describing multiple EE solutions for the diffusive SIS epidemic model.

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DEDICATION

I dedicate this dissertation to the countless lives who, whether through discrimination or violence, were unable to achieve their educational goals.

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CHAPTER 1

INTRODUCTION

Mathematics is a place where you can do things which you can't do in the real world.

-Marcus du Sautoy

Infectious diseases have played a prominent role in the course of human history. From the earliest days of human history, infectious diseases have heavily impacted the structure of society and pushed the boundaries of medicine forward. As shown during the COVID-19 pandemic, these types of diseases can cause massive disruption to our everyday routines. Therefore, it is crucial to be able to come up with a method to help better understand the way an infectious disease will behave once it starts spreading within a population. One way to do this is to use mathematical knowledge and insight to develop a model that aims to represent the spread of a disease.

Early attempts to mathematically model the spread of infectious diseases began with the work of Daniel Bernoulli in [4] in which Bernoulli used mathematics to show the benefits of smallpox inoculation. Later on, the medical doctor Ronald Ross, along with the mathematician Hilda Hudson, developed and studied an ordinary differential equation (ODE) model to better understand why epidemics grow to the scale that they do [26, 27, 28]. Inspired by the work of Ross and Hudson, William Ogilvy Kermack and Anderson Gray McKendrick further developed the ideas of Ross and Hudson to create what we now know as the susceptible-infected-recovered model (SIR model) [17]. It was their results that motivated researchers throughout the years to create new types of infectious disease models.

To seek models that better reflect reality, researchers began building increasingly complex models that consider various factors that might impact the spread of disease such as population movement, advection, spatial heterogeneity, birth and death rates, or even vaccination rates and herd immunity. This dissertation focuses on a diffusive partial differential equation epidemic model that takes into account a moving population within a spatially heterogeneous environment. Our goal is to establish whether disease control strategies that restrict population movement are effective at limiting the spread of disease. Along the way, we discover that the choice of how to represent the method of infection can dramatically change the expected dynamics of the disease and can demonstrate the difficulties in controlling such a disease.

1.1 Developing the Model

Let S be the population of susceptible people. These are the people who have not yet been infected with the disease and, as such, do not carry the disease. For an infectious disease with k strains, let I_i, $i = 1, \ldots, k$, be the population of people infected with strain i of the disease. In our model, the time rate of change of the susceptible population S will be the rate due to random movement plus the rate at which infected people recover from the disease minus the rate due to infection. Similarly, the time rate of change for each infected subpopulation I_i is the rate due to random movement plus the infection rate minus the rate due to recovery. For simplicity, we neglect any birth rates and death rates. These factors will be incorporated in future work. Without considering random movement, this information provides the structure to form the susceptible-infected-susceptible (SIS) model. In fact, this produces an ODE model which can take the form given by

$$\begin{cases} S'(t) = \text{rate of recovery} - \text{rate of infection} \\ I_i'(t) = \text{rate of infection} - \text{rate of recovery} \\ \text{Initial Conditions} \end{cases} \tag{SIS-ODE}$$

As we will see, the dynamics of the ODE model are very simple and can be summarized very nicely in most cases. However, the ODE model has a few limitations. One major limitation is that the ODE model assumes that infection and recovery are spatially independent so that both rates are the same throughout the environment. In actuality, geographic, social, and political differences can produce spatial heterogeneity for both infection and recovery. Therefore, a more realistic infectious disease model should include a spatially heterogeneous environment where infection and recovery rates change depending on location.

Another limitation of the ODE model is that it assumes that the population remains static; there is no sense that the population is moving around. In our increasingly globalized society, people are always on the move traveling from country to country and city to city. As observed during the COVID-19 pandemic, this type of movement can significantly impact the spread of an infectious disease. Hence, an effective model should also take into account the way that populations move within the environment.

Incorporating random population movement turns the ODE model (SIS-ODE) into a partial differential equation (PDE) model. This yields the diffusive SIS model

$$\begin{cases} \partial_t S = \text{rate due to random movement} + \text{rate of recovery} - \text{rate of infection} \\ \partial_t I_i = \text{rate due to random movement} + \text{rate of infection} - \text{rate of recovery} \\ \text{Initial and Boundary Conditions} \end{cases} \tag{SIS-PDE}$$

Here, it is important to note that we also have the choice to include the optional recovered (or removed) group, R. This group consists of people who are infected with the disease, recover from the disease, and obtain immunity from the disease. When the R group is taken into consideration, and assuming that recovered individuals gain total immunity so that they cannot be reinfected, then the model (SIS-ODE) turns into the well-known SIR model of Kermack and McKendrick. However, if the recovered individuals gain only some partial immunity, then the model (SIS-ODE) turns into the equally well-known susceptible-infected-recovered-susceptible (SIRS) model. We might also consider an intermediate group of people E composed of those people who have been exposed to disease and carry the disease but are not yet infectious. When both the R and E groups are considered in the model, then (SIS-ODE) becomes the SEIR model. We do not consider the SEIR model in this dissertation. However, ongoing work is being done to study a diffusive SEIR model and some interesting results have been established.

1.2 Dynamics of the ODE Model

One possible way of representing the model (SIS-ODE) is by the following system of ordinary differential equations

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI + \gamma I, \quad t > 0 \\ \dfrac{dI}{dt} = \beta SI - \gamma I, \qquad t > 0 \\ S(0) + I(0) = N \end{cases} \tag{1.1}$$

where β is the disease transmission rate, γ is the recovery rate, and N is the initial population size. The quantities β, γ, and N are all assumed to be positive. By adding the first two equations of (1.1), one can easily see that $S(t) + I(t) = N$ for all $t \geq 0$. With this in mind, we can solve (1.1) to obtain the solution (for $\beta N - \gamma \neq 0$)

$$\begin{cases} S(t) = \dfrac{1}{1 - Ce^{(\beta N - \gamma)t}} \left(N - C\dfrac{\gamma}{\beta} e^{(\beta N - \gamma)t} \right) \\ I(t) = \dfrac{1}{1 - Ce^{(\beta N - \gamma)t}} \left((\dfrac{\gamma}{\beta} - N) Ce^{(\beta N - \gamma)t} \right) \end{cases} \tag{1.2}$$

where C is some constant. Using the formulas in (1.2), we can observe that $I(t) \to 0$ as $t \to +\infty$ if and only if $\beta N - \gamma < 0$ or $\dfrac{\beta N}{\gamma} < 1$. Moreover, $I(t) \to N - \dfrac{\gamma}{\beta}$ as $t \to +\infty$ if and only if $\beta N - \gamma > 0$ or $\dfrac{\beta N}{\gamma} > 1$. If $\beta N - \gamma = 0$, then the solution to (1.1) is

$$\begin{cases} S(t) = N - \dfrac{1}{\beta t + D} \\ I(t) = \dfrac{1}{\beta t + D} \end{cases} \tag{1.3}$$

where D is some constant. It is easy to see from (1.3) that $I(t) \to 0$ as $t \to +\infty$. Thus, the solutions to (1.1) satisfy $I(t) \to 0$ as $t \to +\infty$ provided that $\dfrac{\beta N}{\gamma} \leq 1$. In other words, we should expect the infectious disease to die off when β, γ, and N satisfy $\dfrac{\beta N}{\gamma} \leq 1$.

The quantity $\dfrac{\beta N}{\gamma}$ is referred to as the *basic reproduction number* for the model (1.1). In epidemic models such as (1.1), the basic reproduction number, often represented by $\mathcal{R}_0$, is a quantity that represents the expected number of secondary infections once you drop a single infected person in a population composed entirely of susceptible people. Therefore, as shown above, the basic reproduction number $\mathcal{R}_0 = \dfrac{\beta N}{\gamma}$ serves as a parameter that determines whether an infectious disease will die out or will persist. This phenomenon is not unique to (1.1) as it is well-known that if $\mathcal{R}_0 < 1$ then the disease is expected to die off while if $\mathcal{R}_0 > 1$ then the disease will persist. With this idea in mind, we want to see if the same idea applies in a more intricate model such as the one given in (SIS-PDE).

1.3 The PDE Model

Incorporating the ideas from (SIS-PDE) and representing each equation as a diffusive partial differential equation, we obtain a model that takes into account the spatial heterogeneity of the environment and the movement of the populations. This is given by

$$\begin{cases} \partial_t S = d_S \Delta S + \sum_{i=1}^{k} \gamma_i I_i - S \sum_{i=1}^{k} \beta_i I_i, & x \in \Omega,\ t > 0, \\ \partial_t I_i = d_i \Delta I_i + \beta_i S I_i - \gamma_i I_i, & x \in \Omega,\ t > 0,\ i = 1, \ldots, k, \\ 0 = \partial_{\vec{n}} S = \partial_{\vec{n}} I_i, & x \in \partial\Omega,\ t > 0,\ i = 1, \ldots, k. \end{cases} \tag{1.4}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary for some positive integer n. The quantities d_S and d_I represent the movement rates of the susceptible and infected populations, respectively. The functions $\beta_i : \overline{\Omega} \to \mathbb{R}$, $i = 1, ..., k$, are Hölder continuous functions representing the infection rate of strain i of the infectious disease while the functions $\gamma_i : \overline{\Omega} \to \mathbb{R}$, $i = 1, ..., k$, are Hölder continuous functions representing the recovery rate of strain i of the infectious disease. In future chapters, the number N will represent the total population size.

This dissertation will focus on the case when $k = 1$. Namely, we study the model given by

$$\begin{cases} \partial_t S = d_S \Delta S + \gamma(x) I - \beta(x) S I, & x \in \Omega,\ t > 0, \\ \partial_t I = d_I \Delta I + \beta(x) S I - \gamma(x) I, & x \in \Omega,\ t > 0, \\ 0 = \partial_{\vec{n}} S = \partial_{\vec{n}} I, & x \in \partial\Omega,\ t > 0. \end{cases} \cdot \tag{1.5}$$

The multiple-strain model (1.4) will be explored in future work.

1.4 Representing Infection

In the diffusive epidemic model (1.5), the term representing the action of infection is given by $\beta(x)SI$. In general, for similar models to (1.5), the term representing the method of infection takes the form of $f(S,I)$ where f is some locally Lipschitz function on $\mathbb{R}_+^2$ satisfying $f(S,0) = 0$ for $S \geq 0$.

One common example used in the literature is $f(S,I) = \dfrac{\beta SI}{S+I}$. This is referred to as the standard incidence transmission mechanism or frequency-dependent transmission mechanism since a disease modeled with this infection term is assumed to depend only on the contact rate between susceptible individuals and infected individuals. Allen et al. in their seminal work [3] studied a model similar to (1.5), but with the standard incidence infection term. They defined a basic reproduction number and showed that their model obeys similar dynamics to (1.1). Inspired by the work of Allen et al., other researchers strove to expand upon the results of [3] (see [8, 9, 14, 22, 23, 24, 25]). Results on the multiple-strain diffusive SIS model with the standard incidence transmission mechanism can be found in [1, 18, 19].

The transmission mechanism used in (1.5), given by $f(S,I) = \beta SI$, is referred to as the mass-action incidence transmission mechanism or density-dependent transmission mechanism. The mass-action transmission mechanism is best used to model a disease that spreads quicker when the population density is higher. The diffusive SIS model with mass-action transmission is not as well-studied as the model with standard incidence. Although there has been progress in this direction (see [10, 31, 30]), there are various gaps in the literature concerning this type of model. The results presented in this dissertation were made in an attempt to close some of these gaps.

1.5 Closing Remarks

The rest of this dissertation is organized as follows. The second chapter collects some important preliminary results that are necessary for the discussion of our main results. The third chapter is dedicated to evaluating the effectiveness of disease control strategies involving the restriction of population movement. In doing so, we examine the well-posedness of the model, as well as the existence and uniqueness (or non-uniqueness) of endemic equilibrium solutions of the model (1.5). We also examine the asymptotic profiles of the endemic equilibrium solutions as the diffusion rates for the susceptible or infected population decay to 0. The fourth chapter demonstrates the striking effect that the inclusion of population movement can have on the dynamics of the disease. In that chapter, we show that there is a precise range of the parameters of the model that leads to a multiplicity of endemic equilibrium solutions. This demonstrates that it can be difficult to predict the dynamics of an infectious disease under certain conditions. In the final chapter, we discuss possible future directions to explore.

CHAPTER 2

NOTATION, DEFINITIONS, AND SOME PRELIMINARY RESULTS

This chapter will be devoted to introducing some of the notation and definitions used throughout the subsequent chapters. It will also contain a few preliminary results necessary for our discussion. Some of these results will be stated without proof, but adequate references will be provided. New results will be supplemented with detailed proofs.

2.1 Notation and Definitions

Throughout this chapter and the subsequent chapters, we suppose that Ω is an open bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Let $C(\overline{\Omega})$ denote the Banach space of uniformly continuous functions on Ω endowed with sup norm

$$\|u\|_\infty := \max_{x\in\overline{\Omega}} |u(x)|, \quad u \in C(\overline{\Omega}).$$

Since we are concerned with population density functions, then we mainly focus on the closed subset of nonnegative continuous functions on $\overline{\Omega}$,

$$C^+(\overline{\Omega}) := \{u \in C(\overline{\Omega}) \mid u \geq 0\}.$$

Given an element $u \in C(\overline{\Omega})$, we introduce the following notation:

$$u_{\min} = \min_{x\in\overline{\Omega}} u(x), \quad u_{\max} := \max_{x\in\overline{\Omega}} u(x), \quad \text{and} \quad \overline{u} := \frac{1}{|\Omega|}\int_\Omega u.$$

Given a real number $r \in \mathbb{R}$, we use the standard notation of $r_+ = \max\{0, r\}$ and $r_- = \max\{0, -r\}$, so that $r = r_+ - r_-$ and $|r| = r_+ + r_-$.

Given $q \in [1, +\infty)$ and an integer $k \geq 1$, let $L^q(\Omega)$ denote the Banach space of L^q-integrable functions on Ω and $W^{k,q}(\Omega)$ denote the usual Sobolev space. In particular, for $q \geq 1$, let

$$W^{2,q}(\Omega) = \{u \in L^q(\Omega) \mid D^\alpha u \in L^q(\Omega), \quad \text{for all } |\alpha| \leq 2\}$$

and

$$W^{2,q}_{\vec{n}}(\Omega) = \{u \in W^{2,q}(\Omega) \mid \partial_{\vec{n}} u = 0 \text{ on } \partial\Omega\}.$$

For each integer $k \geq 1$, define

$$C^k(\Omega) = \{u \in C(\Omega) \mid D^\alpha u \in C(\Omega), \quad \text{for all } |\alpha| \leq k\}.$$

For each $0 < \alpha \leq 1$ and for a given function $u : \Omega \to \mathbb{R}$, we define the seminorm $[u]_\alpha$ by

$$[u]_\alpha := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

We can then define the Banach space $C^{k,\alpha}(\Omega)$ by

$$C^{k,\alpha}(\Omega) := \left\{u \in C^k(\Omega) \mid \sum_{|\alpha|\leq k} \|D^\alpha u\|_\infty + \sum_{|\alpha|=k} [D^\alpha u]_\alpha < +\infty\right\}$$

with norm

$$\|u\|_{C^{k,\alpha}(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_\infty + \sum_{|\alpha|=k} [D^\alpha u]_\alpha.$$

Given $(S_0, I_0) \in [C^+(\overline{\Omega})]^2$, we shall denote by $(S(t,x), I(t,x))$ the unique classical solution to (1.5) with initial data (S_0, I_0) defined on a maximal interval of existence $[0, T_{\max})$. The existence and uniqueness of $(S(t,x), I(t,x))$ will be discussed in the next chapter.

2.2 The Disease-Free Equilibrium (DFE) and the Basic Reproduction Number ($\mathcal{R}_0$).

A time-independent solution, $(S(x), I(x))$, of (1.5) is said to be an *equilibrium solution* of (1.5). In other words, it solves the system

$$\begin{cases} d_S \Delta S + \gamma I - \beta SI = 0, & x \in \Omega, \\ d_I \Delta I - \gamma I + \beta SI = 0, & x \in \Omega, \\ \partial_{\vec{n}} S = \partial_{\vec{n}} I = 0, & x \in \partial\Omega, \\ N = \displaystyle\int_\Omega (S+I). \end{cases} \tag{2.1}$$

A solution of (2.1) which is of the form $(S, 0)$ is known as a disease-free equilibrium (DFE) solution. The only DFE solution of (1.5) when $d_S > 0$ is given by $\left(\frac{N}{|\Omega|}, 0\right)$. To see this, add the first two equations in (2.1) to obtain

$$\partial_t(S + I) = d_S \Delta S + d_I \Delta I.$$

Integrating both sides over Ω, we have

$$\frac{d}{dt}\int_\Omega (S+I) = \int_\Omega (d_S \Delta S + d_I \Delta I).$$

Applying the integration by parts formula to the right-hand-side and recalling that $\partial_{\vec{n}} S = \partial_{\vec{n}} I = 0$, then

$$\frac{d}{dt}\int_\Omega (S+I) = 0.$$

Therefore, $\int_\Omega (S+I) \equiv N$ for some positive constant N. For $I \equiv 0$, we have

$$\begin{cases} \Delta S = 0, & x \in \Omega \\ \partial_{\vec{n}} S = 0, & x \in \partial\Omega. \end{cases}$$

Multiply the first equation by S and integrate by parts to obtain $\int_\Omega |\nabla S|^2 = 0$. Hence, $S \equiv C$ for some constant C. Therefore, $N = \int_\Omega (S+I) = \int_\Omega C = |\Omega| C$. Thus, $(S, I) = \left(\frac{N}{|\Omega|}, 0\right)$ is the unique

DFE solution of (1.5).

To assist in discussing the dynamics of (1.5), we now provide a formula for the basic reproduction number for (1.5). Recall that the basic reproduction number is interpreted to be the expected number of secondary infections if an infected person were dropped into a population composed entirely of susceptible people. Therefore, to obtain a formula for the basic reproduction number, we linearize (1.5) at the DFE solution $\left(\frac{N}{|\Omega|}, 0\right)$ to obtain

$$\begin{cases} \partial_t I = d_I \Delta I - \gamma I + \beta \frac{N}{|\Omega|} I, & x \in \Omega,\ t > 0, \\ \partial_{\vec{n}} I = 0, & x \in \partial\Omega,\ t > 0. \end{cases}$$

After applying the next-generation matrix approach of Diekmann and Heesterbeek (see [11]), it can be shown that the basic reproduction number $\mathcal{R}_0$ is the unique positive number for which there is a positive solution φ to the weighted eigenvalue problem

$$\begin{cases} d_I \Delta \varphi - \gamma \varphi + \frac{1}{\mathcal{R}_0} \frac{N}{|\Omega|} \beta \varphi = 0, & x \in \Omega, \\ \partial_{\vec{n}} \varphi = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Then from (2.2) and using the variational characterization of the principal eigenvalue, we get that $\mathcal{R}_0 = \mathcal{R}(N, d_I)$ is given by

$$\mathcal{R}_0 = \frac{N}{|\Omega|} \mathcal{R}_1 \tag{2.3}$$

where $\mathcal{R}_1 = \mathcal{R}_1(d_I)$ is given by the formula

$$\mathcal{R}_1(d_I) := \sup_{\varphi \in H^2(\Omega) \setminus \{0\}} \frac{\int_\Omega \beta \varphi^2}{\int_\Omega [d_I |\nabla \varphi|^2 + \gamma \varphi^2]}. \tag{2.4}$$

This is the inspiration for why the authors in [10], who also studied the PDE-SIS model given by (1.5), defined its basic reproduction number $\mathcal{R}_0$ by the formula (2.3). We shall let φ_1 denote the positive eigenfunction associated with $\mathcal{R}_0$ satisfying $\|\varphi_1\|_{L^2(\Omega)} = 1$.

Next, we define the local reproduction function by

$$\mathfrak{R}(x) := \frac{N\beta(x)}{|\Omega|\gamma(x)}, \quad x \in \Omega, \tag{2.5}$$

and the sets

$$H^+ := \{x \in \overline{\Omega} \mid \mathfrak{R}(x) > 1\} \quad \text{and} \quad H^- := \{x \in \overline{\Omega} \mid \mathfrak{R}(x) < 1\}.$$

The set H^+ is referred to as the high-risk area for the disease and the set H^- is the low-risk area. The low-risk area, H^-, is the region in $\overline{\Omega}$ where (locally) the disease infection rate is lower than the rate of recovery. The high-risk area, H^+, is the region where the infection rate is higher than the recovery rate. Intuitively, this means that individuals living on H^- are at a lower risk of contracting the disease, while those residing on H^+ are at a higher risk of contracting the disease.

Due to (2.3), since the formula of $\mathcal{R}_0$ is written in terms of $\mathcal{R}_1$, to study the dependence of $\mathcal{R}_0$ with respect to d_I, it is enough to examine the dependence of $\mathcal{R}_1$ with respect to d_I. The following result, due to Allen et al. in [3], collects some important properties of $\mathcal{R}_1$.

Lemma 2.2.1. *[3]*Lemma 2.3*

(i) *If* $\dfrac{\beta}{\gamma}$ *is constant, then* $\mathcal{R}_1 = \dfrac{\beta}{\gamma}$ *for all* $d_I > 0$.

(ii) *If* $\dfrac{\beta}{\gamma}$ *is not constant, then* $\mathcal{R}_1$ *is strictly decreasing in* d_I

$$\lim_{d_I \to +\infty} \mathcal{R}_0(N, d_I) = \frac{N \int_\Omega \beta}{|\Omega| \int_\Omega \gamma} \quad \textit{and} \quad \lim_{d_I \to 0^+} \mathcal{R}_0(N, d_I) = \max_{x \in \overline{\Omega}} \mathfrak{R}(x). \tag{2.6}$$

In particular, if

$$\overline{(\gamma/\beta)} < \overline{\gamma}/\overline{\beta}, \tag{2.7}$$

then $1 < (\overline{\gamma/\beta})\mathcal{R}_1$ *whenever* $0 < d_I < \mathcal{R}_1^{-1}(1/(\overline{\gamma/\beta}))$; $1 = (\overline{\gamma/\beta})\mathcal{R}_1$ *whenever* $d_I = \mathcal{R}_1^{-1}(1/(\overline{\gamma/\beta}))$; *and* $1 > (\overline{\gamma/\beta})\mathcal{R}_1$ *whenever* $d_I > \mathcal{R}_1^{-1}(1/(\overline{\gamma/\beta}))$.

Due to Lemma 2.2.1, we can see that $\mathcal{R}_0$ is nondecreasing with respect to the infected population diffusion rate d_I. It is also clear from (2.3) that $\mathcal{R}_0$ depends linearly on the total population size

N and, in fact, increases with N. It is important to note that $\mathcal{R}_0$ is independent of the susceptible population diffusion rate d_S.

For each $q \in [2, +\infty)$, consider the linear elliptic operator on $L^q(\Omega)$ (resp. on $C(\overline{\Omega})$), defined by

$$\mathcal{L}^* w = d_I \Delta w + (l^* \beta - \gamma) w, \quad w \in \mathrm{Dom}_q \text{ (resp. } w \in \mathrm{Dom}_\infty\text{)}, \tag{2.8}$$

where $\mathrm{Dom}_q = \{u \in W^{2,q}(\Omega) \mid \partial_{\vec{n}} u = 0 \text{ on } \partial\Omega\}$ and $\mathrm{Dom}_\infty = \{u \in \cap_{q \geq 1} \mathrm{Dom}_q \mid \mathcal{L}^* u \in C(\overline{\Omega})\}$. By [12]*Theorem 1, pg. 357, since $\mathcal{L}^*$ is symmetric on $L^2(\Omega)$, then $L^2(\Omega)$ has an orthonormal basis formed by eigenfunctions of $\mathcal{L}^*$. Moreover, by (2.2) and (2.3), φ_1 is an eigenvector of $\mathcal{L}^*$ associated with its principal eigenvalue 0. Furthermore, we can decompose $L^2(\Omega)$ as

$$L^2(\Omega) = \mathrm{span}\{\varphi_1\} \oplus \{w \in L^2(\Omega) \mid \int_\Omega w\varphi_1 = 0\}.$$

Thus,

$$C(\overline{\Omega}) = \mathrm{span}\{\varphi_1\} \oplus \mathcal{Z}_\infty \quad \text{and} \quad L^q(\Omega) = \mathrm{span}\{\varphi_1\} \oplus \mathcal{Z}_q, \quad q \geq 2,$$

where

$$\mathcal{Z}_\infty = \left\{ g \in C(\overline{\Omega}) \mid \int_\Omega g\varphi_1 = 0 \right\}$$

and

$$\mathcal{Z}_q = \left\{ w \in L^q(\Omega) \mid \int_\Omega w\varphi_1 = 0 \right\} \quad \forall\, q \geq 2.$$

Moreover, we have that $\mathcal{L}^*_{|\mathcal{Z}_q \cap \mathrm{Dom}_q} : \mathcal{Z}_q \cap \mathrm{Dom}_q \to \mathcal{Z}_q$ is invertible for each $q \in [2, +\infty]$.

2.3 The Endemic Equilibrium (EE) Problem

An endemic equilibrium (EE) solution is an equilibrium solution where $I > 0$. In epidemic models, the existence of an endemic equilibrium solution indicates the persistence of the disease. It is common in basic epidemic models that existence of an endemic equilibrium solution is related to the magnitude of the basic reproduction number.

To study the existence of an endemic equilibrium solution of (1.5), we find an equivalent problem. This is the essence of our next result.

Lemma 2.3.1. *Suppose that d_S and d_I are fixed positive numbers. The following is true:*

(i) *If $(S(x), I(x))$ is an equilibrium solution of* (1.5)*, then the function*

$$\kappa(x) = d_S S(x) + d_I I(x), \quad x \in \Omega \tag{2.9}$$

is constant. Moreover, if we let

$$\tilde{S} = \frac{S}{\kappa} \quad \text{and} \quad \tilde{I} = \frac{I}{\kappa}, \tag{2.10}$$

then $(\kappa, \tilde{S}, \tilde{I})$ satisfies

$$\tilde{S} = \frac{1}{d_S}(1 - d_I \tilde{I}), \tag{2.11}$$

$$N = \frac{\kappa}{d_S} \int_\Omega ((1 - d_I \tilde{I}) + d_S \tilde{I}) \tag{2.12}$$

and

$$\begin{cases} d_I \Delta \tilde{I} + \left(\frac{\kappa \beta}{d_S}(1 - d_I \tilde{I}) - \gamma \right) \tilde{I}, & x \in \Omega, \\ \partial_{\vec{n}} \tilde{I} = 0, & x \in \partial\Omega, \\ 0 \le \tilde{I} < \frac{1}{d_I}, & x \in \Omega. \end{cases} \tag{2.13}$$

(ii) *If $(\kappa, \tilde{S}, \tilde{I})$ solves* (2.11), (2.12), *and* (2.13)*, then $(S, I) = (\kappa \tilde{S}, \kappa \tilde{I})$ is an equilibrium solution of* (3.1).

Proof. (i) It can be shown that κ satisfies

$$\begin{cases} \Delta \kappa = 0, & x \in \Omega, \\ \partial_{\vec{n}} \kappa = 0, & x \in \partial\Omega. \end{cases}$$

Therefore, we can see that κ is some constant. If we divide both sides of (2.9) by κ and solve for $\tilde{S}$, then we obtain (2.11). Equation (2.12) follows from (2.10) and the fact that $N = \int_\Omega (S + I)$. System (2.13) follows when $S = \kappa \tilde{S} = \frac{\kappa}{d_S}(1 - d_I \tilde{I})$ is substituted in the second equation of (2.1).

(ii) It can be easily verified that, if $(\kappa, \tilde{S}, \tilde{I})$ solves (2.11), (2.12), and (2.13), then $(S, I) = (\kappa\tilde{S}, \kappa\tilde{I})$ is an equilibrium solution of (1.5). $\square$

2.4 Profiles of Solutions to a One Parameter Family of Diffusive-Logistic Equations

Due to Lemma 2.3.1, we can see that the existence of an endemic equilibrium solution of system (1.5) is equivalent to the existence of a positive solution of the logistic equation (2.13). Hence, the current section is devoted to studying the dynamics of positive solutions of (2.13).

Given $f \in C(\overline{\Omega})$ and $d > 0$, let $\lambda(d, f)$ denote the principal eigenvalue of the linear elliptic equation

$$\begin{cases} 0 = d\Delta\varphi + f\varphi + \lambda\varphi & x \in \Omega, \\ 0 = \partial_{\vec{n}}\varphi & x \in \partial\Omega. \end{cases} \tag{2.14}$$

It is well-known that $\lambda(d, f)$ is simple and is given by the variational formula

$$\lambda(d, f) = \min_{\varphi \in W^{1,2}(\Omega)\setminus\{0\}} \frac{d\int_\Omega |\varphi|^2 - \int_\Omega f\varphi^2}{\int_\Omega \varphi^2}. \tag{2.15}$$

Lemma 2.4.1. (i) *If $f \in C(\overline{\Omega})$ is a constant function, then $\lambda(d, f) = -f$ for all $d > 0$.*

(ii) *If $f \in C(\overline{\Omega})$ is not constant, then the mapping $d \mapsto \lambda(d, f)$ is strictly increasing with*

$$\lim_{d\to 0^+} \lambda(d, f) = -f_{\max} \quad \text{and} \quad \lim_{d\to+\infty} \lambda(d, f) = -\frac{1}{|\Omega|}\int_\Omega f.$$

(iii) *If $f_1, f_2 \in C(\overline{\Omega})$ satisfy $f_1 \leq, \neq f_2$, then $\lambda(d, f_2) < \lambda(d, f_1)$ for all $d > 0$.*

For convenience, we introduce the quantity l^*, defined by

$$l^* := \frac{1}{\mathcal{R}_1} \quad \text{so that} \quad \mathcal{R}_0 = \frac{N}{l^*|\Omega|}. \tag{2.16}$$

where $\mathcal{R}_0$ is given by (2.3). From [3], we know that $1 - \mathcal{R}_1$ and $\lambda(d_I, \beta - \gamma)$ have the same sign.

For every positive number $l > 0$, consider the KPP-type elliptic equation

$$\begin{cases} d_I \Delta u + (l\beta(1-d_I u) - \gamma)u = 0, & x \in \Omega, \\ \partial_{\vec{n}} u = 0, & x \in \partial\Omega. \end{cases} \tag{2.17}$$

It is well-known that there exists a unique nonnegative stable solution, denoted by u^l, to (2.17) for every $l > 0$. The next result summarizes some important pieces of information on u^l. This result is essential for the proofs of some of the main results. For convenience, we define the function $\mathcal{N}_{d_I}(l)$ by

$$\mathcal{N}_{d_I}(l) := \frac{l}{l^*|\Omega|}\int_\Omega (1-d_I u^l) = l(\overline{1-d_I u^l})\mathcal{R}_1, \quad l \geq l^*. \tag{2.18}$$

Lemma 2.4.2. *Fix $d_I > 0$ and let $\mathcal{R}_1$ be given by* (2.4)*.*

(i) *The elliptic equation* (2.17) *has a unique positive solution u^l if and only if $l > l^*$. Moreover,*

$$0 < u^l < \frac{1}{d_I}, \tag{2.19}$$

$$\lim_{l \to l^*} \|u^l\|_\infty = 0, \tag{2.20}$$

$$\lim_{l \to +\infty} \left\| u^l - \frac{1}{d_I} \right\|_\infty = 0, \tag{2.21}$$

$$\lim_{l \to +\infty} \left\| l(1-d_I u^l) - \frac{\gamma}{\beta} \right\|_\infty = 0, \tag{2.22}$$

and

$$\lim_{l \to l^*} \left\| \frac{u^l}{l-l^*} - \frac{\mathcal{R}_1 \int_\Omega \beta\varphi_1^2}{d_I \int_\Omega \beta\varphi_1^3}\varphi_1 \right\|_{C^1(\overline{\Omega})} = 0. \tag{2.23}$$

(ii) *The mapping $(l^*, +\infty) \ni l \mapsto u^l \in C^1(\overline{\Omega})$ is smooth and strictly increasing. Setting $v^l = \partial_l u^l$ for every $l > l^*$, then v^l satisfies*

$$\begin{cases} d_I \Delta v^l + (l\beta(1-2d_I u^l) - \gamma)v^l + \beta(1-d_I u^l)u^l = 0, & x \in \Omega, \\ \partial_{\vec{n}} v^l = 0, & x \in \partial\Omega, \end{cases} \tag{2.24}$$

$$\lim_{l \to l^*} \left\| v^l - \frac{\mathcal{R}_1 \int_\Omega \beta\varphi_1^2}{d_I \int_\Omega \beta\varphi_1^3}\varphi_1 \right\|_{C^1(\overline{\Omega})} = 0, \tag{2.25}$$

and

$$\lim_{l\to+\infty}\left\| l^2 v^l - \frac{\gamma}{d_I\beta}\right\|_\infty = 0. \tag{2.26}$$

(iii) *The function* $\mathcal{N}_{d_I}$ *defined by* (2.18) *is continuously differentiable and*

$$\mathcal{N}_{d_I}(l^*) = 1 \quad \textit{and} \quad \lim_{l\to+\infty}\mathcal{N}_{d_I}(l) = (\overline{\gamma/\beta})\mathcal{R}_1. \tag{2.27}$$

Proof of Lemma 2.4.2-(i). (i) **Proof of** (2.19). If we linearize (2.17) at $u \equiv 0$, we obtain the eigenvalue problem

$$\begin{cases} d_I\Delta u + l\beta u - \gamma u + \lambda u = 0, & x\in\Omega,\\ \partial_{\vec{n}}u = 0, & x\in\partial\Omega. \end{cases} \tag{2.28}$$

Recall that that for every $l > 0$, $\lambda(d_I, l\beta-\gamma)$ is the principal eigenvalue of (2.28). Since (2.17) is a logistic equation, then it has (unique) positive solution u^l if and only if $\lambda(d_I, l\beta-\gamma) < 0$ ([5]). By Lemma 2.4.1, we have that $\lambda(d_I, l\beta-\gamma)$ is strictly decreasing in l. Note also from (2.2) that $\lambda(d_I, l^*\beta-\gamma) = 0$. Therefore, $\lambda(d_I, l\beta-\gamma) < 0$ if and only if $l > l^*$. As a result, (2.17) has a (unique) positive solution if and only if $l > l^*$. Moreover, for every $l > l^*$, the unique positive solution u^l of (2.17) is linearly and globally stable. Now, it is easy to see that the constant positive function $u = \frac{1}{d_I}$ is a supersolution of (2.17). Therefore, it since u^l is unique and globally stable, we must have that $0 < u^l < \frac{1}{d_I}$.

Proof of (2.20)**.** Since the solution u^l is uniformly bounded in l by (2.19), then by the regularity theory for elliptic equations, as $l \to l^*$, up to a subsequence, $u^l \to u^*$ uniformly in Ω, where u^* is a nonnegative solution of (2.17). However, since (2.17) has no positive solution for $l = l^*$, then $u^* \equiv 0$. Since $u^* \equiv 0$ is independent of the chosen subsequence, we must have that (2.20) holds.

Proof of (2.21)**.** Let z^l be the function given by $z^l = \frac{1}{d_I} - u^l$. Then, z^l is strictly positive by (2.19) and satisfies

$$\begin{cases} d_I\Delta z + l\beta d_I(z - \frac{\gamma}{l\beta d_I})u^l = 0, & x\in\Omega,\\ \partial_{\vec{n}}z = 0, & x\in\partial\Omega. \end{cases} \tag{2.29}$$

Observe that $\overline{z} = \min_{x\in\Omega} \dfrac{\gamma(x)}{l\beta(x)d_I}$ is a supersolution to (2.29). So we have that

$$0 < z^l = \frac{1}{d_I} - u^l \le \overline{z} = \min_{x\in\Omega} \frac{\gamma(x)}{l\beta(x)d_I}$$

or

$$0 < \left\| u^l - \frac{1}{d_I} \right\|_\infty \le \frac{1}{l} \min_{x\in\Omega} \frac{\gamma(x)}{\beta(x)d_I}.$$

Thus, $\left\| u^l - \dfrac{1}{d_I} \right\|_\infty \to 0$ as $l \to +\infty$.

Proof of (2.22). Observe that the function $w_{d_I,l} = l(1 - d_I u_{d_I,l})$ satisfies

$$\begin{cases} \dfrac{1}{l}\Delta w_{d_I,l} + (\gamma - \beta w_{d_I,l})u_{d_I,l} = 0 & x \in \Omega, \\ \partial_{\vec{n}} w_{d_I,l} = 0 & x \in \partial\Omega. \end{cases} \tag{2.30}$$

Hence, since by (2.21) we have that $u_{d_I,l} \to \dfrac{1}{d_I}$ uniformly in x as $l \to +\infty$, we can make use of the singular perturbation theory [5] to obtain that $w \to \dfrac{\gamma}{\beta}$ uniformly in x as $l \to +\infty$.

Proof of (2.23). If we write u^l as

$$u^l = (l - l^*)\psi^l, \quad l > l^* \tag{2.31}$$

then we need only find the limit of ψ^l as $l \to l^*$. Set

$$c(l) := \int_\Omega \varphi_1 \psi^l \quad \text{and} \quad \tilde{\psi}^l := \psi^l - c(l)\varphi_1, \quad l > l^*. \tag{2.32}$$

Since $\int_\Omega \varphi_1^2 = 1$, then $\int_\Omega \tilde{\psi}^l \varphi_1 = 0$. Hence,

$$\tilde{\psi}^l \in \mathcal{Z}_\infty \quad \text{and} \quad u^l = (l - l^*)(c(l)\varphi_1 + \tilde{\psi}^l), \quad l > l^*. \tag{2.33}$$

Setting $\hat{\psi}^l := \dfrac{\tilde{\psi}^l}{l - l^*}$, then

$$\hat{\psi}^l \in \mathcal{Z}_\infty \quad \text{and} \quad u^l = (l - l^*)(c(l)\varphi_1 + (l - l^*)\hat{\psi}^l), \quad l > l^*. \tag{2.34}$$

We now show the existence of a positive constant K_{d_I} such that

$$\frac{|\Omega|\beta_{\min}\varphi_{1\min}}{ld_I\,\|\beta\|_\infty K_{d_I}} \le c(l) \le \frac{|\Omega|\,\|\beta\|_\infty\,\|\varphi_1\|_\infty}{ld_I\beta_{\min}} \tag{2.35}$$

Multiply (2.2) by u^l and (2.17) by φ_1, integrate the resulting equations, and take their difference to get

$$0 = \int_\Omega (l\beta - ld_I\beta u^l - \gamma)u^l\varphi_1 - \int_\Omega (l^*\beta - \gamma)u^l\varphi_1$$

so that

$$(l-l^*)\int_\Omega \beta u^l\varphi_1 = ld_I\int_\Omega \beta(u^l)^2\varphi_1, \quad l > l^*. \tag{2.36}$$

By (2.36) and Hölder's inequality, we obtain for all $l > l^*$ that

$$\begin{aligned}
(l-l^*)\int_\Omega u^l\varphi_1 &\ge \frac{1}{\|\beta\|_\infty}(l-l^*)\int_\Omega \beta u^l\varphi_1 \\
&= \frac{ld_I}{\|\beta\|_\infty}\int_\Omega \beta(u^l)^2\varphi_1 \\
&= \frac{ld_I}{\|\beta\|_\infty}\int_\Omega (u^l\varphi_1)^2\frac{\beta}{\varphi_1} \\
&\ge \frac{ld_I}{\|\beta\|_\infty}\left(\frac{\beta}{\varphi_1}\right)_{\min}\int_\Omega (u^l\varphi_1)^2 \\
&\ge \frac{ld_I\beta_{\min}}{\|\beta\|_\infty\,\|\varphi_1\|_\infty}\int_\Omega (u^l\varphi_1)^2 \\
&\ge \frac{ld_I\beta_{\min}}{|\Omega|\,\|\beta\|_\infty\,\|\varphi_1\|_\infty}\left(\int_\Omega u^l\varphi_1\right)^2.
\end{aligned}$$

Thus,

$$\frac{|\Omega|\,\|\beta\|_\infty\,\|\varphi_1\|_\infty}{ld_I\beta_{\min}} \ge \frac{1}{l-l^*}\int_\Omega u^l\varphi_1, \quad l > l^*. \tag{2.37}$$

On the other hand, we have from (2.31) and (2.32) that

$$\int_\Omega u^l\varphi_1 = (l-l^*)\int_\Omega \psi^l\varphi_1 = c(l)(l-l^*), \quad l > l^*. \tag{2.38}$$

Hence, by (2.37) and (2.38), we obtain

$$c(l) \le \frac{|\Omega| \, \|\beta\|_\infty \, \|\varphi_1\|_\infty}{l d_I \beta_{\min}}, \quad l > l^*. \tag{2.39}$$

Observe from (2.20) that

$$\lim_{l \to l^*} \left\| l\beta(1 - d_I u^l) - \gamma \right\|_\infty = \|l^* \beta - \gamma\|_\infty . \tag{2.40}$$

Then, from (2.22) and (2.40), we get

$$\sup_{l > l^*} \left\| l\beta(1 - d_I u^l) - \gamma \right\|_\infty < +\infty.$$

Hence, due to Harnack's inequality for elliptic equations and since u^l is a solution to (2.17), that there exists a positive constant K_{d_I}, which is independent of $l > l^*$, such that

$$\left\| u^l \right\|_\infty \le K_{d_I} u^l_{\min}, \quad l > l^*. \tag{2.41}$$

Then, (2.36) and (2.41) together yield that for every $l > l^*$

$$\begin{aligned}
(l - l^*) \int_\Omega u^l \varphi_1 &\le \frac{l - l^*}{\beta_{\min}} \int_\Omega \beta u^l \varphi_1 \\
&= \frac{l d_I}{\beta_{\min}} \int_\Omega \beta (u^l)^2 \varphi_1 \\
&\le \frac{l d_I K_{d_I} \|\beta\|_\infty u^l_{\min}}{\beta_{\min}} \int_\Omega u^l \varphi_1 \\
&= \frac{l d_I K_{d_I} \|\beta\|_\infty}{\beta_{\min} \varphi_{1,\min}} (u^l_{\min} \varphi_{1,\min}) \int_\Omega u^l \varphi_1 \\
&\le \frac{l d_I K_{d_I} \|\beta\|_\infty}{\beta_{\min} \varphi_{1,\min}} \int_\Omega (u^l \varphi_1)^2 \\
&\le \frac{l d_I K_{d_I} \|\beta\|_\infty}{|\Omega| \beta_{\min} \varphi_{1,\min}} \left(\int_\Omega u^l \varphi_1 \right)^2 .
\end{aligned}$$

Combined with (2.38),

$$\frac{|\Omega| \beta_{\min} \varphi_{1,\min}}{l d_I K_{d_I} \|\beta\|_\infty} \le \frac{1}{l - l*} \int_\Omega u^l \varphi_1 = c(l), \quad l > l^*. \tag{2.42}$$

Thus, from (2.39) and (2.42), we get that (2.35) holds.

Next, we will show that

$$\lim_{l\to l^*} c(l) = \frac{\int_\Omega \beta\varphi_1^2}{l^* d_I \int_\Omega \beta\varphi_1^3}. \tag{2.43}$$

Since u^l solves (2.17) and satisfies (2.33), then $\tilde{\psi}^l$ satisfies

$$\begin{cases} d_I\Delta(c(l)\varphi_1 + \tilde{\psi}^l) + (l\beta(1-d_I u^l) - \gamma)(c(l)\varphi_1 + \tilde{\psi}^l) = 0, & x \in \Omega, \\ \partial_{\vec{n}}\tilde{\psi}^l = 0, & x \in \partial\Omega. \end{cases} \tag{2.44}$$

Additionally, since φ_1 satisfies (2.2), then (2.44) can be written as

$$\begin{cases} d_I\Delta\tilde{\psi}^l + (l-l^*)c(l)\beta(1 - ld_I(c(l)\varphi_1 + \tilde{\psi}^l))\varphi_1 + \beta(l(1-d_I u^l) - l^*)\hat{\psi}^l = 0, & x \in \Omega, \\ \partial_{\vec{n}}\hat{\psi}^l = 0, & x \in \partial\Omega. \end{cases} \tag{2.45}$$

Since $\hat{\psi}^l = \dfrac{\tilde{\psi}^l}{l-l^*}$ for $l > l^*$, then from (2.45) we can get

$$\begin{cases} \mathcal{L}^*\hat{\psi}^l + c(l)\beta(1 - ld_I(c(l)\varphi_1 + (l-l^*)\hat{\psi}^l))\varphi_1 + \beta(l(1-d_I u^l) - l^*)\hat{\psi}^l = 0, & x \in \Omega, \\ \partial_{\vec{n}}\hat{\psi}^l = 0, & x \in \partial\Omega. \end{cases}$$

Hence,

$$\hat{\psi}^l = -\mathcal{L}^{*,-1}_{|\mathcal{Z}_q\cap \mathrm{Dom}_q}\Big(c(l)\beta(1 - ld_I(c(l)\varphi_1 + (l-l^*)\hat{\psi}^l))\varphi_1 + \beta(l(1-d_I u^l) - l^*)\hat{\psi}^l\Big), \quad q \geq 2,\ l > l^*.$$

Set $M_q := \left\|\mathcal{L}^{*,-1}_{|\mathcal{Z}_q\cap \mathrm{Dom}_q}\right\|$, then

$$\begin{aligned} \left\|\hat{\psi}^l\right\|_{W^{2,q}(\Omega)} \leq& M_q \left\|c(l)\beta(1 - ld_I(c(l)\varphi_1 + (l-l^*)\hat{\psi}^l))\varphi_1 + \beta(l(1-d_I u^l) - l^*)\hat{\psi}^l\right\|_{L^q(\Omega)} \\ \leq& M_q \|\beta\|_\infty \left((l-l^*)(ld_I c(l)\|\varphi_1\|_\infty + 1) + ld_I\left\|u^l\right\|_\infty\right)\left\|\hat{\psi}^l\right\|_{W^{2,q}(\Omega)} \\ &+ M_q c(l)\|\beta\|_\infty (1 + ld_I c(l)\|\varphi_1\|_\infty)|\Omega|^{1/q}. \end{aligned} \tag{2.46}$$

In view of (2.20) and (2.35), we can see that, for each $q \geq 1$,

$$M_q \|\beta\|_\infty \left((l - l^*)(l d_I c(l) \|\varphi_1\|_\infty + 1) + l d_I \left\| u^l \right\|_\infty \right) \to 0 \quad \text{as} \quad l \to l^*$$

so that, for each $q \geq 2$, there is a $K_q^* > 0$ such that

$$\left\| \hat{\psi}^l \right\|_{W^{2,q}(\Omega)} \leq K_q^*, \quad 0 < l^* < l < l^* + \tilde{\epsilon}_0 \quad \text{for some } \tilde{\epsilon}_0. \tag{2.47}$$

If we choose $q \gg n$ so that $W^{2,q}(\Omega)$ is compactly embedded in $C^1(\overline{\Omega})$, then it follows from (2.20), (2.35), and (2.47) that (after possibly passing to a subsequence) $c(l) \to c^*$ and $\hat{\psi}^l \to \hat{\psi}^*$ in $C^1(\overline{\Omega})$ as $l \to l*$, where c^* is a positive number and $\hat{\psi}^* \in C^2(\Omega) \cap \mathcal{Z}_\infty$ satisfies

$$\begin{cases} -\mathcal{L}^* \hat{\psi} * = c^* \beta (1 - l^* d_I c^* \varphi_1) \varphi_1, & x \in \Omega, \\ \partial_{\vec{n}} \hat{\psi}^* = 0, & x \in \partial\Omega. \end{cases} \tag{2.48}$$

Since $\mathcal{L}^*(\text{Dom}_\infty \cap \mathcal{Z}_\infty) = \mathcal{Z}_\infty \subset \mathcal{Z}_2$, we can multiply (2.48) by φ_1 and integrate over Ω to obtain

$$0 = c^* \int_\Omega \beta (1 - l^* c^* d_I \varphi_1) \varphi_1^2.$$

Hence, since $c^* > 0$, we have that $c^* = \dfrac{\int_\Omega \beta \varphi_1^2}{l^* d_I \int_\Omega \beta \varphi_1^3}$. Since c^* is independent of the chosen subsequence, then we can conclude that $c(l) \to c^*$ as $l \to l^*$. Furthermore, since $\mathcal{L}^*$ is invertible on $\mathcal{Z}_\infty \cap \text{Dom}_\infty$, then $\hat{\psi}^*$ is the unique solution of (2.48) in $\mathcal{Z}_\infty \cap \text{Dom}_\infty$ and $\hat{\psi}^l \to \hat{\psi}^*$ in $C^1(\overline{\Omega})$ as $l \to l^*$. Thus, we have that

$$\frac{u^l}{l - l^*} = \psi^l = c(l)\varphi_1 + (l - l^*)\hat{\psi}^l \to \frac{\int_\Omega \beta \varphi_1^2}{l^* d_I \int_\Omega \beta \varphi_1^3} \varphi_1 \quad \text{as } l \to l^* \quad \text{in } C^1(\overline{\Omega}).$$

□

Proof of Lemma 2.4.2-(ii). Linearizing (2.17) at u^l, we obtain the eigenvalue problem

$$\begin{cases} d_I\Delta\varphi + (l\beta(1-2d_Iu^l)-\gamma)\varphi + \lambda\varphi = 0 & x\in\Omega, \\ \partial_{\vec{n}}\varphi = 0 & x\in\partial\Omega. \end{cases} \tag{2.49}$$

Recall from (2.14) that $\lambda(d_I, l\beta(1-2d_Iu^l)-\gamma)$ is the principal eigenvalue of (2.49). By Lemma 2.4.1-(iii), we have that

$$\begin{aligned}\lambda(d_I, l\beta(1-2d_Iu^l)-\gamma) \geq& \lambda(d_I, ld_I\beta_{\min}u^l_{\min} + l\beta(1-d_Iu^l)-\gamma) \\ \geq& ld_I\beta_{\min}u^l_{\min} + \lambda(d_I, l\beta(1-d_Iu^l)-\gamma) = ld_I\beta_{\min}u^l_{\min} > 0.\end{aligned}$$

Thus, u^l is linearly stable. It then follows from the Implicit Function Theorem that u^l is continuously differentiable in l, with derivative denoted by v^l. An implicit differentiation of (2.17) shows that v^l is the unique solution of (2.24). Finally, since $\beta(1-d_Iu^l)u^l > 0$, it follows from the maximum principle for elliptic equations that $v^l > 0$ on $\overline{\Omega}$. Note that the function on the right-hand-side of (2.17) is analytic in $l > l^*$. Then, the function u^l is also analytic in $l > l^*$ by the Implicit Function Theorem.

Next, we show that (2.25) and (2.26) hold. For each $l > l^*$, let $\psi^l = \dfrac{u^l}{l-l^*} = c(l)\varphi_1 + \tilde{\psi}^l$ be defined as in (2.31). By (2.23), we can see that for any positive number A, we have

$$1 - d_Iu^l - d_IlA\psi^l \to 1 - d_Il^*Ac^*\varphi_1 \quad \text{as } l\to l^* \quad \text{uniformly in } \Omega,$$

where $c^* = \dfrac{\int_\Omega \beta\varphi_1^2}{l^*d_I\int_\Omega\beta\varphi_1^3} > 0$. Hence, we can choose $0 < A_1 < A_2$ so that

$$1 - d_Iu^l - d_IlA_2\psi^l < 0 < 1 - d_Iu^l - d_IlA_1\psi^l, \quad l^* < l < l^* + \epsilon_0$$

for some $\epsilon_0 > 0$. Then, from (2.44), we have that

$$d_I\Delta(A_1\psi^l) + (l\beta(1-2d_Iu^l)-\gamma)(A_1\psi^l) + \beta(1-d_Iu^l)u^l = \beta(1-d_Iu^l-d_IlA_1\psi^l)u^l > 0, \quad x\in\Omega,$$

and

$$d_I\Delta(A_2\psi^l)+(l\beta(1-2d_Iu^l)-\gamma)(A_2\psi^l)+\beta(1-d_Iu^l)u^l=\beta(1-d_Iu^l-d_IlA_2\psi^l)u^l<0,\quad x\in\Omega,$$

for every $l^*<l<l^*+\epsilon_0$. Therefore, by (2.24) and the comparison principle for elliptic equations, we have

$$A_1\psi^l<v^l<A_2\psi^l,\quad 0<l^*<l<l^*+\epsilon_0.$$

Additionally, by (2.24) and estimates for elliptic equations (after possibly passing to a subsequence), there is a strictly positive function $v^*\in C^2(\Omega)$ with $v^l\to v^*$ in $C^1(\overline{\Omega})$ as $l\to l^*$. Moreover, v^* satisfies the system

$$\begin{cases} d_I\Delta v^*+(l^*\beta-\gamma)v^*=0, & x\in\Omega,\\ \partial_{\vec{n}}v^*=0, & x\in\partial\Omega.\end{cases}$$

Hence, we must have that $v^*=c^{**}\varphi_1$ for some positive number c^{**}. Now, multiply (2.24) by ψ^l and multiply (2.44) by v^l, then integrate over Ω. Taking the difference of the resulting equations and using the fact that $u^l=(l-l^*)\psi^l$, we obtain that

$$\int_\Omega\beta(1-d_Iu^l-ld_Iv^l)(\psi^l)^2=0,\quad l>l^*.$$

Letting $l\to l^*$, we get $\int_\Omega\beta(1-d_Il^*c^{**}\varphi_1)(c^*\varphi_1)^2=0$. Solving for c^{**}, we obtain $c^{**}=c^*$, which is independent of the chosen subsequence. Therefore, $v^l\to c^*\varphi_1$ in $C^1(\overline{\Omega})$ as $l\to l^*$ and so we have proven (2.25).

Finally, if we set $p^l=l^2v^l$ for each $l>l^*$, then it follows from (2.24) that p^l satisfies

$$\begin{cases} \frac{d_I}{l}\Delta p^l+\frac{\beta}{l}(z^l-\frac{\gamma}{\beta})p^l+u^l\beta(z^l-d_Ip^l)=0, & x\in\Omega,\\ \partial_{\vec{n}}p^l=0, & x\in\partial\Omega,\end{cases}\tag{2.50}$$

where $z^l=l(1-d_Iu^l)$ for $l>l^*$. Hence, since (from (2.21) and (2.22)) $u^l\to\frac{1}{d_I}$ and $z^l\to\frac{\gamma}{\beta}$ in $C(\overline{\Omega})$ as $l\to+\infty$, then we can utilize the singular perturbation theory for elliptic equations to deduce from (2.50) that $p^l \to \dfrac{\gamma}{d_I\beta}$ as $l \to +\infty$, uniformly on $\overline{\Omega}$. This shows (2.26). □

Proof of Lemma 2.4.2-(iii). The regularity of $\mathcal{N}_{d_I}$ follows from the fact that the map $l \mapsto u^l$ is continuously differentiable. Hence, from (2.20),

$$\mathcal{N}_{d_I}(l^*) = \lim_{l\to l^*}\left(\frac{l}{l^*|\Omega|}\int_\Omega (1-d_I u^l)\right) = 1.$$

Additionally, from (2.22),

$$\lim_{l\to+\infty}\mathcal{N}_{d_I}(l) = \lim_{l\to+\infty}\left(\frac{l}{l^*|\Omega|}\int_\Omega (1-d_I u^l)\right) = \lim_{l\to+\infty}\left(\frac{1}{l^*|\Omega|}\int_\Omega l(1-d_I u^l)\right) = \frac{1}{l^*|\Omega|}\int_\Omega \frac{\gamma}{\beta} = (\overline{\gamma/\beta})\mathcal{R}_1.$$

Thus, we have shown (2.27). □

2.5 Some Abstract Results on Parabolic Equations

Consider the semilinear initial value problem

$$\begin{cases} \partial_t u(t) = A(u(t)) + F(t, u(t)), & t > 0, \\ u(0) = u_0 \end{cases} \tag{2.51}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X and $F : [0, T] \times X \to X$ is continuous in t and Lipschitz in u. We say that u is a *mild solution* of (2.51) if it satisfies the integral equation given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(s, u(s))\, ds.$$

The following result from Pazy guarantees the (local) existence and uniqueness of mild solutions of (2.51).

Theorem 2.5.1. *[21]*Theorem 1.4, pg. 185 Let X be a Banach space and $F : [0, +\infty)\times X \to X$ be continuous in t for $t \geq 0$ and locally Lipschitz continuous in u, uniformly in t on bounded intervals. If A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X, then for every $u_0 \in X$ there* *is a* $t_{\max} \leq +\infty$ *such that the initial value problem*

$$\begin{cases} \partial_t u(t) = A(u(t)) + F(t, u(t)), \quad t \geq 0, \\ u(0) = u_0, \end{cases}$$

has a unique mild solution u *on* $[0, t_{\max})$. *Moreover, if* $t_{\max} < +\infty$, *then*

$$\lim_{t \to t_{\max}^+} \|u(t)\|_\infty = +\infty.$$

The next result, also from Pazy, describes when mild solutions to (2.51) become classical solutions.

Theorem 2.5.2. *[21]*Theorem 1.5, pg. 187 Let* A *be the infinitesimal generator of a* C_0-*semigroup* $\{T(t)\}_{t\geq 0}$ *on* X. *If* $F : [0, T] \times X \to X$ *is continuously differentiable from* $[0, T] \times X$ *into* X, *then the mild solution of* (2.51) *with* $u_0 \in D(A)$ *is a classical solution of the initial value problem.*

CHAPTER 3

SINGLE-STRAIN MODEL: WELL-POSEDNESS OF THE INITIAL VALUE PROBLEM, EXISTENCE AND ASYMPTOTIC PROFILES OF THE ENDEMIC EQUILIBRIUM SOLUTIONS

In this chapter, we evaluate whether a control strategy involving the restriction of population movement is effective at reducing the spread of disease. In this direction, we examine the existence and asymptotic profiles of endemic equilibrium solutions as the movement rates of either the susceptible population or the infected population become increasingly restricted.

3.1 Well-Posedness of the Initial Value Problem

In this section, we will discuss the global existence and uniqueness of classical solutions to (1.5). Recall that if (1.5) has a classical solution, then it should satisfy

$$\begin{cases} \partial_t S = d_S \Delta S + \gamma(x) I - \beta(x) S I, \quad x \in \Omega,\ t > 0, \\ \partial_t I = d_I \Delta I + \beta(x) S I - \gamma(x) I, \quad x \in \Omega,\ t > 0, \\ 0 = \partial_{\vec{n}} S = \partial_{\vec{n}} I \\ N = \displaystyle\int_\Omega (S + I) \end{cases}$$

where we have included the equation for the total population size N. To guarantee that our solutions reflect biological realism, we take our initial conditions to be nonnegative and continuous up to the boundary of $\overline{\Omega}$. That is, we additionally assume the initial data $(S_0, I_0) \in [C^+(\overline{\Omega})]^2$. The following theorem summarizes the existence and uniqueness of classical solutions to (3.1) with this given initial data.

Theorem 3.1.1. *Given initial data $(S_0, I_0) \in [C^+(\overline{\Omega})]r$, there exists a unique classical solution*

$(S(t,x), I(t,x))$ *to* (1.5) *defined for all time. Furthermore, there exists* $M^* > 0$*, independent of the initial data* (S_0, I_0) *such that*

$$\limsup_{t\to+\infty} \|I(t,\cdot)\|_\infty \le M^*, \tag{3.1}$$

$$\liminf_{t\to+\infty} \min_{x\in\overline{\Omega}} S(t,x) \ge \min\{N/|\Omega|, (\gamma/\beta)_{\min}\}, \tag{3.2}$$

and

$$\limsup_{t\to+\infty} \|S(t,\cdot)\|_\infty \le \max\{N/|\Omega|, (\gamma/\beta)_{\max}\}. \tag{3.3}$$

Remark 3.1.1. *It is important to note that the existence, uniqueness, and global boundedness of solutions to the initial value problem* (1.5) *was established in [10]. However, the a priori estimates* (3.2) *and* (3.3) *are new to the best of our knowledge.*

3.2 Existence of Endemic Equilibrium Solutions

Next, we discuss the existence and nonexistence of endemic equilibrium solutions of (1.5).

Theorem 3.2.1. *Fix* $d_I > 0$ *and* $d_S > 0$*. Then, there is a number* $\mathcal{R}_* \ge 1$*, independent of* N *and satisfying* $\mathcal{R}_* = 1$ *if either* $\dfrac{\gamma}{\beta}$ *is constant or* $d_S \ge d_I$*, such that the following conclusions hold:*

(i) *If* $\mathcal{R}_0 > 1$*, then system* (1.5) *has a finite number* $m \ge 1$ *of endemic equilibrium solutions, with* $m = 1$ *when* $\mathcal{R}_0 > \mathcal{R}_*$*. Moreover, if* $m \ge 2$*, then the endemic equilibrium solutions of* (1.5) *can be totally ordered* $(S_i, I_i), \cdots, (S_m, I_m)$ *in such a way that* $I_i < I_{i+1}$ *on* $\overline{\Omega}$ *for each* $i = 1, \cdots .m-1$*. If, in addition, either* $\dfrac{\gamma}{\beta}$ *is constant or* $d_S \ge d_I$*, then the unique endemic equilibrium solution of* (1.5) *is nondegenerate.*

(ii) *If* $\mathcal{R}_0 \le \max\left\{(\gamma/\beta)_{\min}\mathcal{R}_1, \min\left\{1, \dfrac{d_S}{d_I}\right\}\right\}$*, then system* (1.5) *has no endemic equilibrium solution.*

Remark 3.2.1. *We note that existence and nonexistence of endemic equilibrium solutions was also considered in [10]. However, the finiteness of the set of endemic equilibrium solutions and the fact that this set is totally ordered in the* I*-components seem to be new results. The fact that the unique endemic equilibrium solution of* (1.5) *is nondegenerate when* $d_S > d_I$ *and* $\mathcal{R}_0 > 1$ *also seems to be a new result. Note that if* $\dfrac{\gamma}{\beta}$ *is constant, then* $\mathcal{R}_1 = \dfrac{\beta}{\gamma}$*. Hence, if either* $\dfrac{\gamma}{\beta}$ *is constant or* $d_S \ge d_I$*,* *it follows from Theorem 3.2.1 that* (1.5) *has a (unique) endemic equilibrium solution if and only if* $\mathcal{R}_0 > 1$. *Hence, if either* $\frac{\gamma}{\beta}$ *is constant or* $d_S \geq d_I$, *Theorem 3.2.1 recovers [10]*Theorem 3.11.*

3.3 Asymptotic Profiles of Endemic Equilibrium Solutions

We now discuss the asymptotic limit of endemic equilibrium solutions when we fix a positive diffusion rate d_I of the infected group but allow the diffusion rate of the susceptible group d_S to decay to 0.

Theorem 3.3.1. *Let* $d_I > 0$ *be fixed and assume that* $\mathcal{R}_0 > 1$. *The following are true:*

(i) *If* $N < \int_\Omega \frac{\gamma}{\beta}$, *then there exists a* $C > 0$ *such that every endemic equilibrium solution* (S, I) *of* (1.5) *with* $d_S > 0$ *satisfies*

$$\frac{1}{C} \leq \frac{I}{d_S} \leq C, \quad 0 < d_S < 1. \tag{3.4}$$

Furthermore, up to a subsequence,

$$S \to S^*(\cdot, d_I) := \frac{N(1 - d_I \tilde{I}^*)}{\int_\Omega (1 - d_I \tilde{I}^*)} \quad as \quad d_S \to 0 \tag{3.5}$$

in $C^2(\Omega)$ *where* $0 < \tilde{I}^* < \frac{1}{d_I}$ *is a positive solution of the nonlocal elliptic equation*

$$\begin{cases} d_I \Delta \tilde{I}^* + \left(\frac{N\beta}{\int_\Omega (1 - d_I \tilde{I}^*)} (1 - d_I \tilde{I}^*) - \gamma \right) \tilde{I}^* = 0, & x \in \Omega, \\ \partial_{\vec{n}} \tilde{I}^* = 0, & x \in \partial\Omega. \end{cases} \tag{3.6}$$

(ii) *If* $N > \int_\Omega \frac{\gamma}{\beta}$, *then* (1.5) *has endemic equilibrium solutions* (S, I) *for* $0 < d_S \ll 1$ *satisfying*

$$(S, I) \to \left(\frac{\gamma}{\beta}, \frac{1}{|\Omega|} \left(N - \int_\Omega \frac{\gamma}{\beta} \right) \right) \quad as \quad d_S \to 0 \tag{3.7}$$

uniformly on $\overline{\Omega}$.

Remark 3.3.1. (i) *Theorem 3.3.1 (i) gives the precise decay rate of the infected population at* *endemic equilibrium when* $N < \int_\Omega \frac{\gamma}{\beta}$ *as* $d_S \to 0$. *This implies that if* $N < \int_\Omega \frac{\gamma}{\beta}$ *then the magnitude of the infected group at endemic equilibrium decays to 0 linearly in* d_S *as* d_S *tends to 0. It is also good to note that, in [31], it was proven that the* I*-component of endemic equilibrium solutions will converge to 0 if* $\mathcal{R}_0 > 1$ *and* $N < \int_\Omega \frac{\gamma}{\beta}$. *Theorem 3.3.1 (i) also shows that the magnitude of the* I*-component of endemic equilibrium solutions will be proportional to the diffusion rate,* d_S*, of the susceptible group as* d_S *approaches 0. Furthermore, when we fix the infected group's diffusion rate,* d_I*, the function* $S^*(\cdot, d_I)$ *provides another way of characterizing of the limit profile of the* S*-component of endemic equilibrium solutions as* $d_S \to 0$. *The asymptotic profiles of* $S^*(\cdot, d_I)$ *as* $d_I \to 0$ *will be discussed in Theorem 3.3.3.*

(ii) *We note that Theorem 3.3.1 (ii) answers the conjecture that was made in [31]. If it is additionally assumed that either* $\beta \in C^1(\overline{\Omega})$ *and* $N > \int_\Omega \frac{\gamma}{\beta} + \frac{1}{4}\int_\Omega \frac{|\nabla\beta|^2}{\beta^3}$ *or* $H^+ = \overline{\Omega}$, *the authors in [30] and [31] obtained the same conclusion as in Theorem 3.3.1 (ii). They also conjectured that the same result is true in general. We observe that these results are improved in Theorem 3.3.1 (ii) without requiring other technical assumptions on the infection rate,* β.

The next results show the asymptotic behavior of endemic equilibrium solutions when d_I decays to 0 with either $\frac{d_I}{d_S} \to 0$ or $\frac{d_S}{d_I} \to 0$.

Theorem 3.3.2. *Suppose that* H^+ *is nonempty. Then:*

(i) *There exists* $0 < d_0 \ll 1$ *such that* (1.5) *has a unique endemic equilibrium solution* (S, I) *for every* $0 < d_I \le \min\{d_0, d_S\}$. *Moreover,* (S, I) *satisfies*

$$\lim_{\max\{d_I, \frac{d_I}{d_S}\}\to 0} \left\| S - \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)} \right\|_\infty = 0 \quad \textit{and} \quad \lim_{\max\{d_I, \frac{d_I}{d_S}\}} \int_\Omega I = N - |\Omega| \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}. \tag{3.8}$$

Furthermore, if we define the set Ω^* *by*

$$\Omega^* := \left\{ x \in \overline{\Omega} \mid \frac{\gamma(x)}{\beta(x)} = \min_{y\in\overline{\Omega}} \frac{\gamma(y)}{\beta(y)} \right\}, \tag{3.9}$$

then

$$\lim_{\max\{d_I, \frac{d_I}{d_S}\}\to 0} \max_{x\in K} I(x) = 0 \quad \textit{for any compact subset } K \subset \Omega \setminus \Omega^* \tag{3.10}$$

and there is a (Radon) probability measure μ *on* $\overline{\Omega}$ *such that, up to a subsequence,*

$$I \to \left[N - |\Omega| \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}\right]\mu \quad \textit{weakly}^* \textit{ as } \max\{d_I, \frac{d_I}{d_S}\} \to 0 \tag{3.11}$$

and

$$\mu(\Omega^*) = 1. \tag{3.12}$$

In particular, if Ω^* *is a single point, then* μ *is a Dirac measure.*

(ii) *If* $N < \int_\Omega \frac{\gamma}{\beta}$, *then there is* $0 < d_0 \ll 1$ *such that* (1.5) *has an endemic equilibrium solution* (S, I) *for every* $d_S > 0$ *and* $0 < d_I \le d_0$. *Moreover, there is a positive constant* C *such that* (S, I) *satisfies*

$$\frac{1}{C}\frac{d_S}{d_I} \le \|I\|_\infty \le C\frac{d_S}{d_I}, \tag{3.13}$$

for all d_S, d_I *satisfying* $\max\{d_I, \frac{d_S}{d_I}\} \le d_0$ *and*

$$S(x) \to S_{\nu^*}(x) := \nu^*\left(1 - \left(1 - \frac{\gamma}{\nu^*\beta}\right)_+\right) = \min\{\nu^*, \frac{\gamma(x)}{\beta(x)}\} \quad \textit{as } \max\{d_I, \frac{d_S}{d_I}\} \to 0 \tag{3.14}$$

uniformly on $\overline{\Omega}$, *where* ν^* *is the unique positive number satisfying*

$$\begin{aligned} &\max\{\frac{N}{|\Omega|}, \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}\} < \nu^* < \max_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)} \quad \textit{and} \\ &N = \nu * \int_\Omega (1 - (1 - \frac{\gamma}{\nu^*\beta})_+) = \int_\Omega \min\{\nu^*, \frac{\gamma}{\beta}\}. \end{aligned} \tag{3.15}$$

(iii) *If* $N > \int_\Omega \frac{\gamma}{\beta}$, *then there is* $0 < d_0 \ll 1$ *such that for every* $0 < d_I < d_0$ *there is* $d_{S,I} > 0$ *satisfying* $\lim_{d_I\to 0} \frac{d_{S,I}}{d_I} = 0$ *such that* (1.5) *has endemic equilibrium solutions* (S, I) *with* $d_S = d_{S,I}$ *satisfying*

$$\lim_{d_I\to 0}(S, I) = \left(\frac{\gamma}{\beta}, \frac{1}{|\Omega|}\left(N - \int_\Omega \frac{\gamma}{\beta}\right)\right) \tag{3.16}$$

uniformly on $\overline{\Omega}$.

The next result gives the asymptotic profile of the functions $S^*(\cdot, d_I)$ in (3.5), which can be obtained as a direct consequence of Theorem 3.3.2 (ii), .

Theorem 3.3.3. *Suppose that the set H^+ is nonempty and that the quantity N satisfies $N < \int_\Omega \frac{\gamma}{\beta}$. Then there is $d_0 > 0$ such that $\mathcal{R}_0 > 1$ for every $0 < d_I < d_0$. Furthermore, for every $0 < d_I < d_0$, the function $S^*(\cdot, d_I)$ defined by (3.5) satisfies*

$$\lim_{d_I \to 0} \left\| S^*(\cdot, d_I) - \min\{\nu^*, \frac{\gamma}{\beta}\} \right\|_\infty = 0$$

where ν^* is given by (3.15).

Remark 3.3.2. *Assume that H^+ is nonempty. We make some remarks on Theorem 3.3.2.*

(i) *We begin by noting that the inequality given by $S(x) \geq \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}$ holds for any endemic equilibrium solution $(S(x), I(x))$ of (3.1). Therefore, we have that $\int_\Omega I \leq N - |\Omega| \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}$ for any endemic equilibrium solution $(S(x), I(x))$ of system (3.1). Hence, Theorem 3.3.2 (i) tells us that the local size and total size of the susceptible group at endemic equilibrium will be minimized when the movement rate of the infected group is taken to be sufficiently smaller when compared to the movement rate of the susceptible group. At the same time, the infected group attains its maximal total size.*

(ii) *When $N < \int_\Omega \frac{\gamma}{\beta}$, Theorem 3.3.2* (ii) *demonstrates that the disease may be driven to extinction if the diffusion rate of the susceptible group is considerably reduced compared to the diffusion rate of the infected group. Moreover, since $\nu^* > \frac{N}{|\Omega|}$ by (3.15), it holds that $\overline{H^-} \subset int\left(\left\{\nu^* \geq \frac{\gamma(x)}{\beta(x)}\right\}\right)$. Thus, (3.13) shows that as $\max\left\{d_I, \frac{d_S}{d_I}\right\}$ becomes sufficiently small, the local size of the susceptible group is maximized in the low-risk area as well as in some portions of the high-risk area. These results imply that significantly lowering the diffusion rate of the susceptible group compared to that of the infected group is an effective control strategy that leads to the eradication of the disease.*

(iii) *When $N > \int_\Omega \frac{\gamma}{\beta}$, Theorem 3.3.2 (iii) suggests that the disease may still persist even when significantly reducing the movement rate of the susceptible group compared to the movement rate of the infected group. In this situation, the local population size of the susceptible group is still maximized, as shown by (3.16).*

We emphasize again that the asymptotic profiles of endemic equilibrium solutions as $\max\{d_I, \frac{d_I}{d_S}\}$ tends to 0 was left open in [31]. It is important to point out that [30] partially answers this question in the case where the spatial dimension is $n = 1$ and where it is assumed that $H^+ = \overline{\Omega}$, $d_S > 0$ is fixed, and d_I decays to 0. Despite these stronger assumptions, the results found in [30] do not provide precise knowledge on the limit of endemic equilibrium solutions (S, I) as d_I approaches 0 for a fixed $d_S > 0$. Thus, Theorem 3.3.2 marks a significant improvement on these known results and ultimately answers the open question found in [31] concerning the asymptotic behavior of endemic equilibrium solutions as $\max\left\{d_I, \frac{d_I}{d_S}\right\}$ goes to 0.

3.4 Total Lockdown Scenario

Next, we examine the behavior of equilibrium solutions of (1.5) in a scenario modeling the complete lockdown of the susceptible group and only the susceptible group. In other words, we examine the case of $d_S = 0$ and $d_I > 0$.

Theorem 3.4.1. *Suppose that $d_S = 0$ and $d_I > 0$.*

(i) *If $N \leq \int_\Omega \frac{\gamma}{\beta}$, then (1.5) has:*

(1) *disease-free equilibrium solutions, given by the collection of pairs $(S, I) = (S^*, 0)$ where S^* is any nonnegative, continuous function on $\overline{\Omega}$ with $\int_\Omega S^* = N$.*

(2) *no endemic equilibrium solution.*

Furthermore, if $N < \int_\Omega \frac{\gamma}{\beta}$, then, given any classical solution $(S(t,x), I(t,x))$ of (1.5) with positive initial data, there is $S^* \in C^+(\overline{\Omega})$ with $\int_\Omega S^* = N$ such that

$$(S(t,x), I(t,x)) \to (S^*(x), 0) \quad \text{as } t \to +\infty$$

uniformly on $\overline{\Omega}$.

(ii) *If $N > \int_\Omega \frac{\gamma}{\beta}$, in addition to the disease-free equilibrium solutions described in (i), (1.5) has a unique endemic equilibrium solution* $(S_e^*, I_e^*) := \left(\frac{\gamma}{\beta}, \frac{1}{|\Omega|}\left(N - \int_\Omega \frac{\gamma}{\beta}\right)\right)$.

3.5 Discussion

In this chapter, we investigated whether a reduction in the movement of the population is an effective disease control strategy by examining the asymptotic profiles of endemic equilibrium solutions of the diffusive epidemic model given by (1.5). We specifically examined the cases where either the diffusion rate of the susceptible group, d_S, or the diffusion rate of the infected group, d_I, decay to 0. Finally, we examined the large-time behavior of classical solutions of (1.5) when there is a complete restriction on the movement of only the susceptible group (i.e. $d_S = 0$) while the infected individuals are free to move (i.e. $d_I > 0$).

When the susceptible and infected groups are both free to move, Theorem 3.2.1 demonstrates that the population size is a significant factor in the persistence of the disease. In fact, if we observe that the basic reproduction number $\mathcal{R}_0 = \frac{N}{|\Omega|}\mathcal{R}_1$ is bigger than 1 if and only if $N > \frac{|\Omega|}{\mathcal{R}_1}$, then the quantity $\frac{|\Omega|}{\mathcal{R}_1}$ is a threshold number for the total population size N. In other words, if N is larger than this number, then the disease will persist and become endemic. When N is smaller than $\frac{|\Omega|}{\mathcal{R}_1}$, then there is a possibility for the disease to be controlled. Because of Lemma 2.3.1, it is known that when there is a high-risk area in the environment and when the diffusion rate d_I of the infected individuals is sufficiently small, then we always have that N exceeds the threshold quantity $\frac{|\Omega|}{\mathcal{R}_1}$. Thus, in this case, the disease becomes endemic. Therefore, reducing the diffusion rate of the infected population in an environment containing a nonempty high-risk area could lead to disease persistence.

Because we also want to understand how a reduction in only the movement rates of the susceptible population might affect the dynamics of the disease, we fixed the diffusion rate of the infected group d_I in Theorem 3.3.1 and examined the asymptotic profiles of the endemic equilibrium solutions as $d_S \to 0$. Our results show a minimization of the size of the infected group at endemic equilibrium when the movement rate of the susceptible people becomes sufficiently small. In fact, we can see that, when the overall population size is kept below a new threshold number $\int_\Omega \frac{\gamma}{\beta}$, the total size of the infected population at endemic equilibrium decays at a rate proportional to the diffusion rate of the susceptible group. This shows that, when we only lower the diffusion rate of the susceptible population and when we maintain the overall population size under some critical value, then the impact of the disease on the population may be significantly reduced even in the case where the disease persists. However, if the total size of the population exceeds this new threshold number, then a reduction of only the movement rate of the susceptible group might still lead to the persistence of the disease. However, the size of the infected population at endemic equilibrium will be minimized. The results in Theorem 3.3.2 (ii) and (iii) confirm that these conclusions are still valid even when there is a reduction in the diffusion rate of the infected population. However, the rate of reduction must be sufficiently large compared to that of the susceptible group.

When the infected diffusion rate is lowered at a sufficiently smaller rate than the susceptible diffusion rate, Theorem 3.3.1 (i) shows that, at endemic equilibrium, the infected population will maximize their total size. Moreover, they will concentrate on the region in the disease high-risk area where the susceptible people minimize their local size. At the same time, the size of the susceptible population is minimized uniformly over the entire environment. Additionally, this suggests that restricting the population movement rate is an effective control strategy provided that the movement rate of the susceptible population is kept sufficiently small compared to the movement rate of the infected population.

Lastly, we studied the large-time behavior of solutions of (1.5) in the case of a total lockdown of only the susceptible group. This is represented by setting $d_S = 0$ and $d_I > 0$. In this case, Theorem 3.4.1 predicts that the disease will be eradicated in the long run when $N < \int_\Omega \frac{\gamma}{\beta}$. On the other hand, the disease will persist if we reverse the inequality. We point out that $N < \int_\Omega \frac{\gamma}{\beta}$ implies that the low-risk area H^- of the disease is nonempty. We also point out that there is also the possibility that both $N > \int_\Omega$ and H^- is nonempty. Hence, Theorem 3.3.3 indicates that the creation of a low-risk area is not enough to eventually eradicate the disease when only the susceptible movement rate is completely restricted. This deviates from the prediction of extinction of the disease using the PDE-SIS model based on the standard incidence transmission mechanism.

CHAPTER 4

MULTIPLICITY OF ENDEMIC EQUILIBRIA

The current chapter establishes the existence of multiple endemic equilibrium solutions of the diffusive SIS epidemic model (1.5). From an application point of view, the results obtained here strongly highlight how population movement and spatial heterogeneity can complicate disease dynamics. This implies that it might be difficult to successfully implement disease control strategies.

To emphasize the spatial heterogeneity of the habitat, we shall always suppose, throughout the chapter, that the following standing assumption holds.

(A) The function $\frac{\beta}{\gamma}$ is not constant.

Hence, since **(A)** holds, it follows from Lemma 2.2.1 that $\mathcal{R}_1 = \mathcal{R}_1(d_I)$, introduced in (2.4), is strictly decreasing in d_I. As such, it has an inverse function which we will denote by $\mathcal{R}_1^{-1}$. Recall that the basic reproduction number $\mathcal{R}_0 = \mathcal{R}_0(N, d_I)$ of (1.5) defined by

$$\mathcal{R}_0 = \frac{N}{|\Omega|}\mathcal{R}_1. \tag{4.1}$$

We can observe from (4.1) that $\mathcal{R}_0$ is strictly increasing with respect to N and is independent of d_S. Moreover, if d_I, β, and γ are fixed, then we can vary $\mathcal{R}_0$ from 0 to $+\infty$. This will often be the case in the statement of our results.

4.1 Multiplicity/Uniqueness of Endemic Equilibrium Solutions of System (1.5)

Theorem 4.1.1 (Multiplicity of Endemic Equilibrium Solutions)**.** *Fix $d_I > 0$. Then there exists $\mathcal{R}_0^{low} = \mathcal{R}_0^{low}(d_I)$ satisfying $0 < \mathcal{R}_0^{low} \le \min\{1, (\overline{\gamma/\beta})\mathcal{R}_1\}$ such that the following conclusions hold.*

(i) *If $\mathcal{R}_0 \le \mathcal{R}_0^{low}$, then* (1.5) *has no endemic equilibrium solution for every $d_S > 0$.*

(ii) *If $\mathcal{R}_0 > \mathcal{R}_0^{low}$, then there is $d_1^* = d_1^*(\mathcal{R}_0, d_I) > 0$ such that* (1.5) *has an endemic equilibrium* *solution* (S_{high}, I_{high}) *for every* $0 < d_S < d_1^*$. *Moreover, any other endemic equilibrium solution* (S, I), *if one exists, must satisfy*

$$I(x) < I_{high}(x), \quad x \in \overline{\Omega}. \tag{4.2}$$

(iii) *If* $\mathcal{R}_0^{low} < 1$ *and* $\mathcal{R}_0^{low} < \mathcal{R}_0 < 1$, *then system* (1.5) *has an endemic equilibrium solution* (S_{low}, I_{low}) *for every* $0 < d_S < d_1^*$, *where* d_1^* *is as in* (ii), *satisfying*

$$I_{low}(x) < I_{high}(x), \quad x \in \overline{\Omega}, \tag{4.3}$$

such that any other endemic equilibrium solution (S, I) *of* (1.5), *if one exists, must satisfy*

$$I_{low}(x) < I(x), \quad x \in \overline{\Omega}. \tag{4.4}$$

Let d_I and $\mathcal{R}_0^{\text{low}}$ be given as in Theorem 4.1.1. It can be concluded from Theorem 4.1.1-(i) and (ii) that the quantity $\mathcal{R}_0^{\text{low}}$ serves as a sharp critical number that the basic reproduction number must surpass in order to guarantee the existence of endemic equilibrium solutions of system (1.5) over some range of the susceptible population diffusion rate and the total population size. The asymptotic profiles of the endemic equilibrium solutions of Theorem 4.1.1 as $d_S \to 0$ will be given in Theorem 4.3.1.

It is important to know sufficient conditions for when $\mathcal{R}_0^{\text{low}} < 1$. In this direction, we have:

Proposition 4.1.2. *Suppose that*

$$\overline{\gamma/\beta} < \overline{\gamma}/\overline{\beta}. \tag{4.5}$$

Then $\mathcal{R}_0^{low} < 1$ *for every* $d_I > \mathcal{R}_1^{-1}(\overline{\gamma/\beta})$.

Note that (4.5) holds when $\gamma = \beta^2$ and is not constant. Under the assumption (4.5), we can observe from the previous theorem that there is a range of parameters satisfying $\mathcal{R}_0 < 1$ where (1.5) has at least two endemic equilibrium solutions when the susceptible diffusion rate d_S is small. An immediate question is to know whether (1.5) may have multiple endemic equilibrium solutions for large values of d_S. First, we establish the following result on the uniqueness of endemic equilibrium solutions of (1.5).

Theorem 4.1.3 (Uniqueness of Endemic Equilibrium Solutions)**.** *For every $d_I > 0$, there exists $d_{low} = d_{low}(d_I)$ satisfying $0 \leq d_{low} < d_I$ such that the following hold:*

(i) *If $d_S > d_{low}$ and $\mathcal{R}_0 > 1$, then (1.5) has a unique endemic equilibrium solution.*

(ii) *If $d_S > d_{low}$ and $\mathcal{R}_0 \leq 1$, then (1.5) has no endemic equilibrium solution.*

By Theorem 4.1.3, $\mathcal{R}_0$ is enough to predict the existence of endemic equilibrium solutions of (1.5) for large values of d_S. Note that $d_{\text{low}}(d_I)$ is independent of N and strictly less than d_I. Then, for $d_{\text{low}}(d_I) < d_S < d_I$, system (1.5) has a (unique) endemic equilibrium solution if and only if $\mathcal{R}_0 > 1$. This improves previously known results on the uniqueness of endemic equilibrium solutions of (1.5).

Remark 4.1.1. *We note that if $d_{low}(d_I) > 0$, then for every $0 < d_S < d_{low}(d_I)$, system (1.5) has at least two endemic equilibrium solutions for a range of $\mathcal{R}_0$. This shows that $d_{low}(d_I)$ is a sharp critical number that d_S must exceed to guarantee the uniqueness of endemic equilibrium solutions of (1.5).*

4.2 Bifurcation Curves of Endemic Equilibrium Solutions

The next result complements Theorem 4.1.3 by identifying sufficient conditions that lead to a backward bifurcation curve of endemic equilibrium solutions at $\mathcal{R}_0 = 1$.

Theorem 4.2.1 (Backward Bifurcation Curve)**.** *Fix $d_I > 0$ and suppose that*

$$\overline{\beta\varphi_1^3} < (\overline{\varphi_1})(\overline{\beta\varphi_1^2}). \tag{4.6}$$

Then there is $d_2^* = d_2^*(d_I) > 0$ such that for every $0 < d_S < d_2^*$, as $\mathcal{R}_0$ increases from 0 to $+\infty$, the endemic equilibrium solutions of (1.5) form an unbounded simple connected curve which bifurcates from the set of disease-free equilibrium solutions from the left at $\mathcal{R}_0 = 1$.

Proposition 4.5.3 bellow gives an example of parameters satisfying (4.6). Our next result complements Theorem 4.2.1 again by identifying sufficient conditions that lead to a forward S-shaped bifurcation curve of endemic equilibrium solutions at $\mathcal{R}_0 = 1$.

Theorem 4.2.2 (Forward and S-shaped Bifurcation Curve)**.** *Fix $d_I > 0$ and suppose that*

$$(\overline{\gamma/\beta})\mathcal{R}_1 < 1 \quad \text{and} \quad \overline{\beta\varphi_1^3} > (\overline{\varphi_1})(\overline{\beta\varphi_1^2}). \tag{4.7}$$

Then there is $d_3^* = d_3^*(d_I) > 0$ such that for every $0 < d_S < d_3^*$, as $\mathcal{R}_0$ increases from 0 to $+\infty$, the endemic equilibrium solutions of (1.5) *form an unbounded simple connected curve which bifurcates from the set of disease-free equilibrium solutions from the right at $\mathcal{R}_0 = 1$. Moreover, for every $0 < d_S < d_3^*$, there exist $\mathcal{R}_{0,1}^{d_S} < 1 < \mathcal{R}_{0,2}^{d_S} \leq \mathcal{R}_{0,3}^{d_S}$ such that:*

(i) *If $\mathcal{R}_0 < \mathcal{R}_{0,1}^{d_S}$, then* (1.5) *has no endemic equilibrium solution.*

(ii) *If $\mathcal{R}_0 = \mathcal{R}_{0,1}^{d_S}$, then* (1.5) *has at least one endemic equilibrium solution.*

(iii) *If either $\mathcal{R}_{0,1}^{d_S} < \mathcal{R}_0 \leq 1$ or $\mathcal{R}_0 = \mathcal{R}_{0,2}^{d_S}$, then* (1.5) *has at least two endemic equilibrium solutions.*

(iv) *If $1 < \mathcal{R} < \mathcal{R}_{0,2}^{d_S}$, then* (1.5) *has at least three endemic equilibrium solutions.*

(v) *If $\mathcal{R}_0 > \mathcal{R}_{0,2}^{d_S}$, then* (1.5) *has at least one endemic equilibrium solution. This solution is unique if $\mathcal{R}_0 > \mathcal{R}_{0,3}^{d_S}$.*

Furthermore, $\mathcal{R}_{0,i}^{d_S}$ is strictly increasing in d_S for each $i = 1, 2, 3$ and, as $d_S \to 0$, $\mathcal{R}_{0,1}^{d_S} \to \mathcal{R}_0^{low}$ and $\mathcal{R}_{0,i}^{d_S} \to \mathcal{R}_{0,i}^*$, $i = 2, 3$, for some positive numbers $1 < \mathcal{R}_{0,2}^* \leq \mathcal{R}_{0,3}^*$.

Proposition 4.5.3 below gives an example of parameters for which (4.7) holds.

4.3 Asymptotic Profiles of Endemic Equilibrium Solutions for Small d_S

Next, we explore how the multiplicity of endemic equilibrium solutions of (1.5) may affect disease control strategy. As such, we complement Theorems 4.1.1, 4.1.3, 4.2.1 and 4.2.2 with a result on the asymptotic profiles of endemic equilibrium solutions as $d_S \to 0$. Our result indicates that the disease will either persist or die out depending on how d_S is lowered. More precisely, we have the following result:

Theorem 4.3.1. *Fix $d_I > 0$ and suppose that $\mathcal{R}_0^{low} < 1$, where $\mathcal{R}_0^{low}$ is given by Theorem 4.1.1.*

(i) *Fix $\mathcal{R}_0^{low} < \mathcal{R}_0 < 1$ and let d_1^* be as in Theorem 4.1.1 such that the system (1.5) has a maximal endemic equilibrium solution (S_{high}, I_{high}) and a minimal endemic equilibrium solution (S_{low}, I_{low}) for every $0 < d_S < d_1^*$.*

(i-1) *If $\mathcal{R}_0 < (\overline{\gamma/\beta})\mathcal{R}_1$, then there is a positive constant $C > 0$ such that*

$$\frac{d_S}{C} \leq I_{low} < I_{high} \leq C d_S, \quad 0 < d_S < \frac{d_1^*}{2}, \tag{4.8}$$

$$S_{high} \to S_{high}^* := \frac{N(1 - d_I u_{high}^*)}{\int_\Omega (1 - d_I u_{high}^*)}, \tag{4.9}$$

and

$$S_{low} \to S_{low}^* := \frac{N(1 - d_I u_{low}^*)}{\int_\Omega (1 - d_I u_{low}^*)} \quad \text{as} \quad d_S \to 0 \tag{4.10}$$

in $C^1(\overline{\Omega})$ where $0 < u_{low}^* < u_{high}^* < \dfrac{1}{d_I}$ are classical solutions of the nonlocal elliptic equation

$$\begin{cases} d_I \Delta u^* + \left(\beta \dfrac{\mathcal{R}_0 (1 - d_I u^*)}{\mathcal{R}_1 \int_\Omega (1 - d_I u^*)} - \gamma\right) u^* = 0, & x \in \Omega, \\ \partial_{\vec{n}} u^* = 0, & x \in \partial\Omega. \end{cases} \tag{4.11}$$

(i-2) *If $\mathcal{R}_0 > (\overline{\gamma/\beta})\mathcal{R}_1$, then*

$$\lim_{d_S \to 0^+} \left[\left\| S_{high} - \frac{\gamma}{\beta} \right\|_\infty + \left\| I_{high} - \left(\frac{\mathcal{R}_0}{\mathcal{R}_1} - (\overline{\gamma/\beta}) \right) \right\|_\infty \right] = 0 \tag{4.12}$$

and (S_{low}, I_{low}) satisfies (4.8) and (4.10).

(ii) *In addition, suppose that (4.7) holds. Let d_3^* and $\mathcal{R}_{0,2}^*$ be given by Theorem 4.2.2 and fix $1 < \mathcal{R}_0 < \mathcal{R}_{0,2}^*$. Then, for each $0 < d_S < d_3^*$, system (1.5) has a maximal endemic equilibrium* *solution* (S_{high}, I_{high}) *and a minimal endemic equilibrium solution* (S_{low}, I_{low})*. Moreover, as* $d_S \to 0$*,* I_{low} *satisfies* (4.8)*,* S_{low} *satisfies* (4.10) *up to a subsequence, and* (S_{high}, I_{high}) *satisfies* (4.12)*.*

Theorem 4.2.2 shows that as $d_S \to 0$, the profiles of the endemic equilibrium solutions of (1.5) depends on the chosen subsequence. These results also show that the two scenarios discussed in [31] are possible. Theorem 4.2.2 also complements Theorem 3.3.2 which establishes the asymptotic profiles of endemic equilibrium solutions of (1.5) as $d_I \to 0$.

4.4 Discussion

In this chapter, we examined the questions of multiplicity or uniqueness of the endemic equilibrium solutions of a diffusive epidemic model with the mass-action transmission mechanism. In doing so, we have obtained some interesting results. In particular, we have discovered new phenomena which cannot be observed from either the ODE model (1.1) or the corresponding PDE model (see [3]) with the frequency-dependent transmission mechanism.

As mentioned in the introduction, for the ODE-SIS model with simple nonlinearity (i.e. frequency-dependent or mass-action transmission), the basic reproduction number is enough to completely characterize the existence of endemic equilibrium solutions. This is also the case for the diffusive epidemic model with the frequency-dependent transmission. However, for the diffusive model (1.5) with mass-action transmission, we showed that the basic reproduction number is not enough to predict the persistence of the disease.

Indeed, Theorem 4.1.1 indicates that, for the dynamics of solutions of (1.5), the disease may persist even if $\mathcal{R}_0 < 1$. More precisely, there is a critical number $0 < \mathcal{R}_0^{\text{low}} \leq 1$, uniquely determined by d_I, such that (1.5) has no endemic equilibrium solution if $\mathcal{R}_0 \leq \mathcal{R}_0^{\text{low}}$ for any diffusion rate d_S of the susceptible population. However, thanks to Theorem 4.1.1, if $\mathcal{R}_0^{\text{low}} < \mathcal{R}_0 < 1$, then there exist at least two endemic equilibrium solutions when d_S is sufficiently small. In this case, we see that $\mathcal{R}_0$ is not enough to predict the persistence of the disease. In Proposition 4.1.2, we showed that if the average of the ratio of the recovery rate over the transmission rate is smaller than the ratio of the average of the recovery rate to the average of the transmission rate, then $\mathcal{R}_0^{\text{low}} < 1$ for large values of d_I. By Theorem 4.2.1, if the average of $\beta\varphi_1^3$ is smaller than the product of the averages of φ_1 and $\beta\varphi_1^2$, we again have that $\mathcal{R}_0^{\text{low}} < 1$. In fact, there is a backward bifurcation at $\mathcal{R}_0 = 1$ in this case.

When $\mathcal{R}_0 > 1$, Theorem 4.2.2 shows that it is possible for (1.5) to have at least three endemic equilibrium solutions when d_S is small. Despite the many possible scenarios shown above concerning the global structure of the endemic equilibrium solutions of (1.5), Theorem 4.1.3 shows that $\mathcal{R}_0$ is enough to predict the persistence of the disease if the susceptible population moves at a fast enough rate compared to the infected population.

Thus, from the above results and by Theorem 4.3.1, we can conclude that decreasing the movement rate of the susceptible population could be an effective disease control strategy as long as any accumulation of the population is also minimized.

4.5 Construction of Examples

We conclude this chapter with a construction of examples that satisfy the hypotheses of our main results.

Let $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_m \le \ldots$ denote the eigenvalues of

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, & x \in \Omega, \\ \partial_{\vec{n}}\varphi = 0, & x \in \partial\Omega, \end{cases}$$

satisfying $\lambda_m \to +\infty$ as $m \to +\infty$. Let $\{\varphi_m\}_{m\ge 0}$ denote the orthonormal basis of $L^2(\Omega)$ where each φ_m is an eigenfunction associated with the eigenvalue λ_m for each $m \ge 0$. Now, consider the Banach space $\tilde{\mathcal{Z}} = \{w \in L^2(\Omega) \mid \overline{w} = 0\} = \text{span}(\varphi_0)$. For every $q \ge 2$, the restriction of the Laplace operator on $\text{Dom}_q \cap \tilde{\mathcal{Z}}$ to $\tilde{\mathcal{Z}}_q := L^q(\Omega) \cap \tilde{\mathcal{Z}}$ is invertible. Let $T = \Delta|_{\text{Dom}_q \cap \tilde{\mathcal{Z}}}$ and let $C_q^* := \|T^{-1}\|$. Then for every $w \in \tilde{\mathcal{Z}}_q$, the unique solution $W \in \text{Dom}_q \cap \tilde{\mathcal{Z}}$ of

$$\begin{cases} \Delta W + w = 0, & x \in \Omega \\ \partial_{\vec{n}} W = 0, & x \in \partial\Omega, \\ \overline{W} = 0, \end{cases} \tag{4.13}$$

satisfies

$$\|W\|_{W^{2,q}(\Omega)} \leq C_q^* \|w\|_{L^q(\Omega)} . \tag{4.14}$$

Fix a non-constant, Hölder continuous function h on $\overline{\Omega}$ and fix positive constants k and d_I. Define

$$l_0^* = k, \quad \tilde{\varphi}_0 = \frac{1}{|\Omega|}, \quad \text{and} \quad l_1^* = \overline{h}. \tag{4.15}$$

Let $\tilde{\varphi}_1$ be the unique solution to (4.13) with $w_1 := l_0^*(l_1^* - h)\tilde{\varphi}_0/d_I$. Note that $\tilde{\varphi}_1$ is well-defined since $\int_\Omega w_1 = 0$. Next, define

$$l_2^* = (\overline{(h - l_1^*)h} + k|\Omega|\overline{(h - l_1^*)\tilde{\varphi}_1})/k, \tag{4.16}$$

and $\tilde{\varphi}_2$ to be the unique solution of (4.13) with $w_2 := (kl_2^*\tilde{\varphi}_0 - h(h-l_1^*)\tilde{\varphi}_0 - k(h-l_1^*)\tilde{\varphi}_1)/d_I$. Note also that $\tilde{\varphi}_2$ is well-defined since $\int_\Omega w_2 = 0$. Throughout the rest of this section, whenever h, k, and d_I are given, we shall suppose that $l_0^*, l_1^*, l_2^*, \tilde{\varphi}_0, \tilde{\varphi}_1$, and $\tilde{\varphi}_2$ are defined as above.

Proposition 4.5.1. *Fix $k > 0$ and $d_I > 0$ and suppose that $h = c_m\varphi_m$ for some $m \geq 1$ where c_m is a nonzero constant. Then $\tilde{\varphi}_1 = -(kh)/(|\Omega| d_I \lambda_m)$ and $l_2^* = (1 - k^2/(d_I\lambda_m))\overline{h^2}/k$.*

Proof. It can be verified by inspection. □

Proposition 4.5.2. *Fix $d_I > 0$, $k > 0$, and a Hölder continuous, non-constant function h on $\overline{\Omega}$. For every, $0 < \epsilon < \epsilon_{k,h} := \dfrac{k}{\|h\|_\infty}$, define*

$$\beta_{k,h,\epsilon} := k + \epsilon h \tag{4.17}$$

and let $l^*(\epsilon)$ denote the principal eigenvalue of the weighted linear elliptic equation

$$\begin{cases} d_I\Delta\tilde{\varphi} + \beta_{k,h,\epsilon}(l(\epsilon) - \beta_{k,h,\epsilon})\tilde{\varphi} = 0, & x \in \Omega, \\ \partial_{\vec{n}}\tilde{\varphi} = 0, & x \in \partial\Omega. \end{cases} \tag{4.18}$$

Denote by $\tilde\varphi(\cdot;\epsilon)$ *the unique positive solution of* (4.13) *satisfying* $\int_\Omega \tilde\varphi(\cdot;\epsilon)=1$. *Next, define*

$$\tilde\varphi_3(\cdot;\epsilon)=(\tilde\varphi(\cdot;\epsilon)-\tilde\varphi_0-\epsilon\tilde\varphi_1-\epsilon^2\tilde\varphi_2)/\epsilon^3 \quad \textit{and} \quad l_3^*(\epsilon)=(l^*(\epsilon)-l_0^*-\epsilon l_1^*-\epsilon^2 l_2^*)/\epsilon^3. \tag{4.19}$$

It holds that

$$\limsup_{\epsilon\to0}\|\tilde\varphi_3(\cdot;\epsilon)\|_{C^1(\overline{\Omega})}<+\infty \quad \textit{and} \quad \limsup_{\epsilon\to0}|l_3^*(\epsilon)|<+\infty. \tag{4.20}$$

Proof. By straightforward calculation, we have that $\tilde\varphi_3$ and l_3^* satisfy

$$\begin{cases} d_I\Delta\tilde\varphi_3+(kl_2^*+h(l_1^*-h))(\tilde\varphi_1+\epsilon\tilde\varphi_2+\epsilon^2\tilde\varphi_3) & \\ +(\beta_{k,h,\epsilon}l_3^*+hl_2^*)\tilde\varphi+k(l_1^*-h)(\tilde\varphi_2+\epsilon\tilde\varphi_3)=0, & x\in\Omega,\\ \partial_{\vec n}\tilde\varphi_3=0, & x\in\partial\Omega,\\ \int_\Omega\tilde\varphi_3=0. \end{cases} \tag{4.21}$$

Fix $q\gg1$ such that $W^{2,q}(\Omega)$ is continuously embedded in $C^1(\overline{\Omega})$. Applying similar arguments as in the proof of (2.46), we obtain the existence of $0<\epsilon_{q,k,h}\ll1$ and $M_{q,k,h}>0$ such that

$$\|\tilde\varphi_3(\cdot;\epsilon)\|_{W^{2,1}(\Omega)}\le M_{q,k,h}(1+|l_3^*|), \quad 0<\epsilon\le\epsilon_{q,k,h}. \tag{4.22}$$

After integrating the equation in (4.21), setting $A_1=kl_2^*+h(l_1^*-h)$ and $A_2=k(l_1^*-h)$, and rearranging the terms, we obtain

$$l_3^*\int_\Omega\beta_{k,h,\epsilon}\tilde\varphi=-\int_\Omega A_1(\tilde\varphi_1+\epsilon\tilde\varphi_2)-l_2^*\int_\Omega h\tilde\varphi-\int_\Omega A_2\tilde\varphi_2-\epsilon\int_\Omega(A_2+\epsilon A_1)\tilde\varphi_3.$$

Therefore, from (4.22), and after setting $B=|\Omega|\,\|A_1\|_\infty(\|\tilde\varphi_1\|_\infty+\|\tilde\varphi_2\|_\infty)+\sum_{i=1}^{2}\|A_i\|_\infty+\sum_{i=1}^{3}\|\tilde\varphi_i\|_\infty$, we have, for every $0<\epsilon<\epsilon_{q,k,h}$, that

$$\begin{aligned} |l_3^*(\epsilon)|\int_\Omega\beta_{k,h,\epsilon}\tilde\varphi &\le |\Omega|\,\|A_1\|_\infty(\|\tilde\varphi_1\|_\infty+\epsilon\|\tilde\varphi_2\|_\infty)+|\Omega||l_2^*|\,\|\tilde\varphi\|_\infty+\epsilon(\|A_1\|_\infty+\epsilon\|A_2\|_\infty)\|\tilde\varphi_3\|_\infty\\ &\le B+|\Omega||l_2^*|(\|\tilde\varphi_0\|_\infty+\epsilon\|\tilde\varphi_1\|_\infty+\epsilon^2\|\tilde\varphi_2\|_\infty+\epsilon^3\|\tilde\varphi_3\|_\infty)+\epsilon B\|\tilde\varphi_3\|_\infty \end{aligned}$$

$$\leq (1+|\Omega||l_2^*|)B + \epsilon M_{q,k,k}(B+|\Omega||l_2*|)(1+|l_3^*(\epsilon)|)$$

or, equivalently,

$$|l_3^*|\left(\int_\Omega \beta_{k,h,\epsilon}\tilde{\varphi} - \epsilon M_{q,k,h}(B+|\Omega||l_2^*|)\right) \leq (1+|\Omega||l_2^*|)B + \epsilon M_{q,k,h}(B+|\Omega||l_2^*|), \quad 0<\epsilon<\epsilon_{q,h,k}. \tag{4.23}$$

Observe that

$$\int_\Omega \beta_{k,h,\epsilon}\tilde{\varphi} \geq (k-\epsilon\|h\|_\infty)\int_\Omega \tilde{\varphi} = k - \epsilon\|h\|_\infty, \quad 0<\epsilon<\frac{k}{\|h\|_\infty}.$$

Then, from (4.23), we have that $\limsup_{\epsilon\to 0}|l_3^*| \leq \dfrac{(1+|\Omega||l_2^*|)B}{k}$ which implies, with (4.22), that $\limsup_{\epsilon\to 0}\|\tilde{\varphi}_3\|_{C^1(\Omega)} < +\infty$ since $W^{2,q}(\Omega)$ is continuously embedded in $C^2(\overline{\Omega})$. □

Proposition 4.5.3. *Fix $k>0$ and $d_I>0$ and suppose that $h = c_m\varphi_m$ for some $m \geq 1$. For every $0<\epsilon \ll 1$, let $\gamma_{k,h,\epsilon} = \beta_{k,h,\epsilon}^2$ where $\beta_{h,k,\epsilon}$ is defined by (4.17). Then $\mathcal{R}_1 = \dfrac{1}{l^*(\epsilon)}$. Moreover, for sufficiently small values of ϵ, it holds that*

$$1/\mathcal{R}_1 - \overline{\gamma_{k,h,\epsilon}/\beta_{k,h,\epsilon}} \begin{cases} >0 & \text{if } d_I\lambda_m > k^2, \\ <0 & \text{if } d_I\lambda_m < k^2, \end{cases} \tag{4.24}$$

and

$$\overline{\beta_{h,k,\epsilon}\varphi_1^3} - \overline{\varphi_1}(\overline{\beta_{k,h,\epsilon}\varphi_1^2}) \begin{cases} >0 & \text{if } d_I\lambda_m < 2k^2, \\ <0 & \text{if } d_I\lambda > 2k^2. \end{cases} \tag{4.25}$$

Therefore,

(i) *if $k^2 < d_I\lambda_m < 2k^2$, then $1 > (\overline{\gamma_{k,h,\epsilon}/\beta_{k,h,\epsilon}})\mathcal{R}_1$ and $\overline{\beta_{h,k,\epsilon}\varphi_1^3} > \overline{\varphi_1}(\overline{\beta_{k,h,\epsilon}\varphi_1^2})$ for $0<\epsilon\ll 1$.*

(ii) *if $2k^2 < d_I\lambda_m$, then $\overline{\beta_{h,k,\epsilon}\varphi_1^3} < \overline{\varphi_1}(\overline{\beta_{k,h,\epsilon}\varphi_1^2})$ for $0<\epsilon\ll 1$.*

Proof. From (2.2) and (4.18), we can see that $l^*(\epsilon) = 1/\mathcal{R}_1$. By Proposition 4.5.2, we can write $l^*(\epsilon)$ as

$$1/\mathcal{R}_1 = l^*(\epsilon) = k + \epsilon\overline{h} + \epsilon^2 l_2^* + \epsilon^3 l_3^*(\epsilon), \quad 0<\epsilon\ll 1, \tag{4.26}$$

where l_2^* is given by (4.16) and $l_3^*(\epsilon)$ satisfies (4.20). Since $\overline{\gamma_{k,h,\epsilon}/\beta_{k,h,\epsilon}} = \overline{\beta_{k,h,\epsilon}} = \overline{k+\epsilon h} = k + \epsilon\overline{h}$, we can use Proposition 4.5.1 to get

$$\Big(1/\mathcal{R}_1 - \overline{\gamma_{k,h,\epsilon}/\beta_{k,h,\epsilon}}\Big)/\epsilon^2 = l_2^* - \epsilon l_3^*(\epsilon) = \bigg(\Big(1 - k^2/(d_I\lambda_m)\Big)\overline{h^2}\bigg)/k - \epsilon l_3^*(\epsilon), \quad 0 < \epsilon \ll 1. \tag{4.27}$$

Notice that $\lim_{\epsilon\to 0} \epsilon l_3^*(\epsilon) = 0$ by (4.20) and so (4.24) holds.

Next, we show that (4.25) holds. Set $\tilde{\varphi} = \varphi_1/\Big(\int_\Omega \varphi_1\Big)$. Then

$$\overline{\beta_{k,h,\epsilon}\varphi_1^3} - \overline{\varphi_1}(\overline{\beta_{k,h,\epsilon}\varphi_1^2}) = (\overline{\beta_{k,h,\epsilon}\tilde{\varphi}^3} - \overline{\beta_{k,h,\epsilon}\tilde{\varphi}^2})\, \|\varphi_1\|_{L^1(\Omega)}^3 . \tag{4.28}$$

Observe from (2.2) and the fact that $l^*(\epsilon) = 1/\mathcal{R}_1$ and $\int_\Omega \tilde{\varphi} = 1$, that $\tilde{\varphi}$ is the unique solution of (4.18). By Proposition 4.14, $\tilde{\varphi}$ can be written as

$$\tilde{\varphi} = 1/|\Omega| + \epsilon\tilde{\varphi}_1 + \epsilon^2\tilde{\varphi}_2 + \epsilon^3\tilde{\varphi}_3, \quad 0 < \epsilon \ll 1, \tag{4.29}$$

where $\tilde{\varphi}_3$ satisfies (4.20). Hence,

$$\overline{\beta_{k,h,\epsilon}\tilde{\varphi}^3} - \overline{\beta_{k,h,\epsilon}\tilde{\varphi}^2} = \epsilon \int_\Omega \beta_{k,h,\epsilon}\tilde{\varphi}^2(\tilde{\varphi}_1 + \epsilon\tilde{\varphi}_2 + \epsilon^2\tilde{\varphi}_3). \tag{4.30}$$

For convenience, set $P = \tilde{\varphi}_2 + \epsilon\tilde{\varphi}_3$ and $Q = \tilde{\varphi}_1 + \epsilon P$. Then, $\tilde{\varphi} = 1/|\Omega| + \epsilon Q$ and

$$\beta_{k,h,\epsilon}\tilde{\varphi}^2 Q = \frac{k}{|\Omega|^2}\tilde{\varphi}_1 + \epsilon\bigg(\frac{2k}{|\Omega|}\tilde{\varphi}_1^2 + k\tilde{\varphi}^2 P + h\tilde{\varphi}^2 Q\bigg) + k\epsilon^2\bigg(\frac{2}{|\Omega|}P + Q^2\bigg)\tilde{\varphi}_1.$$

Now, observe that

$$k\tilde{\varphi}^2 P = \frac{k}{|\Omega|^2}\tilde{\varphi}_2 + k\epsilon\bigg(\frac{\tilde{\varphi}_3}{|\Omega|^2} + \bigg(\frac{2}{|\Omega|} + \epsilon Q\bigg)QP\bigg)$$

and

$$h\tilde{\varphi}^2 Q = \frac{1}{|\Omega|^2}h\tilde{\varphi}_1 + \epsilon h\bigg(\frac{P}{|\Omega|^2} + \bigg(\frac{2}{|\Omega|} + \epsilon Q\bigg)Q^2\bigg).$$

Then,

$$\beta_{k,h,\epsilon}\tilde{\varphi}^2 Q = \frac{k}{|\Omega|^2}\tilde{\varphi}_1 + \frac{\epsilon}{|\Omega|^2}\Big(2k|\Omega|\tilde{\varphi}_1^2 + k\tilde{\varphi}_2 + h\tilde{\varphi}_1\Big) + \epsilon^2\tilde{\mathbb{H}}(\cdot;\epsilon),$$

where

$$\tilde{\mathbb{H}}(\cdot;\epsilon) := k\left(\frac{2}{|\Omega|}P + Q^2\right)\tilde{\varphi}_1 + k\left(\frac{\tilde{\varphi}_3}{|\Omega|^2} + \left(\frac{2}{|\Omega|} + \epsilon Q\right)QP\right) + h\left(\frac{P}{|\Omega|^2} + \left(\frac{2}{|\Omega|} + \epsilon Q\right)Q^2\right).$$

Therefore, since $\int_\Omega \tilde{\varphi}_i = 0$, $i = 1, 2$, then we have

$$\frac{1}{\epsilon}\int_\Omega \beta_{k,h,\epsilon}\tilde{\varphi}^2 Q = \frac{1}{|\Omega|^2}\left(2k|\Omega|\int_\Omega \tilde{\varphi}_1^2 + \int_\Omega h\tilde{\varphi}_1\right) + \epsilon\int_\Omega \tilde{\mathbb{H}}(\cdot;\epsilon), \quad 0 < \epsilon \ll 1.$$

Due to Proposition 4.5.1, we have

$$2k|\Omega|\int_\Omega \tilde{\varphi}_1^2 + \int_\Omega h\tilde{\varphi}_1 = \left(\frac{2k^2}{d_I\lambda_m} - 1\right)\frac{k}{|\Omega|d_I\lambda_m}\int_\Omega h^2. \tag{4.31}$$

Thus, since by (4.20) we get $\epsilon\left\|\tilde{\mathbb{H}}(\cdot;\epsilon)\right\|_\infty \to 0$ as $\epsilon \to 0$, then from (4.28), (4.30), and (4.31), we get that (4.25) holds. □

CHAPTER 5

PROOFS OF MAIN RESULTS

This chapter contains the proofs of the results stated in Chapters 3 and 4.

5.1 Proofs of Results from Chapter 3

5.1.1 Proof of Theorem 3.1.1

Proof of Theorem 3.1.1. Let $(S_0, I_0) \in [C^+(\overline{\Omega})]^2$. The proof is divided into three steps.

Step 1. In this step, we show that (1.5) subject to the initial condition (S_0, I_0) has a unique classical solution $(S(t,x), I(t,x))$ defined for all $t \geq 0$. To this end, write (1.5) in the form

$$\begin{cases} \partial_t u = Au + F(u) \\ \partial_{\vec{n}} u = 0 \end{cases}$$

where $u = (S, I)$, $Au = (d_S \Delta S, d_I \Delta I)$, and $F(u) = (\gamma I - \beta SI, \beta SI - \gamma I)$. Observe that A has domain

$$D(A) = \{(S, I) \in \cap_{p \geq 1} W^{2,p}_{\vec{n}}(\Omega) \times W^{2,p}_{\vec{n}}(\Omega) \mid (\Delta S, \Delta I) \in [C(\overline{\Omega})]^2\}.$$

From the theory of parabolic equations (see [21]), A generates an analytic C_0-semigroup $\{T(t)\}_{t \geq 0}$ on $[C(\overline{\Omega})]^2$. That is, given any initial data $u_0 = u(0, \cdot) \in [C(\overline{\Omega})]^2$, then

$$u(t, \cdot; u_0) = T(t)u_0, \quad t > 0,$$

is the classical solution of

$$\begin{cases} \partial_t U = AU, \quad t > 0, \\ U(0, \cdot) = u_0, \end{cases}$$

and $u(t, \cdot; u_0) \in D(A)$ for all $t > 0$. By Theorem 2.5.1, there exists a unique mild solution that exists locally on some time interval $[0, t_{\max})$. By Theorem 2.5.2, this mild solution is a classical solution of (1.5).

If we can show that the local solution is bounded, then the solution is defined for all positive times, in other words, $t_{\max} = +\infty$. To that end, we show that

$$\sup_{t \geq 0} \|u(t)\|_{\infty} < +\infty.$$

We begin by showing that I is nonnegative on $[0, t_{\max})$. Take $a(t,x) = \gamma(x) - \beta(x)S(t,x)$. Then the second equation of (1.5) becomes

$$\begin{cases} \partial_t I = d_I \Delta I + a(t,x)I, & x \in \Omega,\ 0 < t < t_{\max}, \\ \partial_{\vec{n}} I = 0, & x \in \partial\Omega,\ 0 < t < t_{\max}. \end{cases} \tag{5.1}$$

Observe that $\underline{I}(t,x) \equiv 0$ is also a solution of (5.1). Since $I(0,x) \geq 0$ by assumption, we can apply the comparison principle for linear parabolic equations to conclude that $I(t,x) \geq 0$ for all $0 \leq t < t_{\max}$ and $x \in \overline{\Omega}$.

We now show that $S(t,x) \geq 0$ for all $0 \leq t < t_{\max}$. Since $I(t,x) \geq 0$ and $\gamma(x) \geq 0$, then from (1.5) we have that S is a supersolution to

$$\begin{cases} \partial_t W = d_S \Delta W - \beta(x) I W, & x \in \Omega,\ 0 < t < t_{\max}, \\ \partial_{\vec{n}} W = 0, & x \in \partial\Omega,\ 0 < t < t_{\max}. \end{cases} \tag{5.2}$$

Observe that $\underline{S}(t,x) \equiv 0$ is a sub-solution of (5.2). Since $S(0,x) \geq 0$, then after applying the comparison principle again, we get that $S(t,x) \geq 0$ for all $0 \leq t < t_{\max}$ and $x \in \overline{\Omega}$.

Now, from (1.5), we have that S satisfies

$$\begin{cases} \partial_t S = d_S \Delta S + \beta(\frac{\gamma}{\beta} - S)I, & x \in \Omega,\ 0 < t < t_{\max}, \\ \partial_{\vec{n}} S = 0, & x \in \partial\Omega,\ 0 < t < t_{\max}. \end{cases} \tag{5.3}$$

Take $\overline{S} \equiv \left\| \frac{\gamma}{\beta} \right\|_{\infty} + \|S(0,\cdot)\|_{\infty}$. Then, $\overline{S}$ is a super-solution of (5.3) and, since $S(0,x) \leq \overline{S}$, then

$S(t,x) \le \overline{S}$ for all $0 < t < t_{\max}$. Hence, S is uniformly bounded on $(0, t_{\max})$.

Now, since $|S(t,x)| \le \overline{S}$ for all $0 < t < t_{\max}$ and $I(t,x) \ge 0$ for all $t \in (0, t_{\max})$, then

$$\begin{cases} \partial_t I \le d_I \Delta I + \|\beta\|_\infty \overline{S} I & x \in \Omega,\ 0 < t < t_{\max}, \\ \partial_{\vec{n}} I = 0 & x \in \partial\Omega,\ 0 < t < t_{\max}. \end{cases}$$

It then follows from the comparison principle for parabolic equations that

$$\|I(t,\cdot)\|_\infty \le \|I_0\|_\infty e^{t\|\beta\|_\infty \overline{S}}, \quad 0 < t < t_{\max}.$$

This along with the boundedness of $\|S(t,\cdot)\|_\infty$ and Theorem 2.5.1 implies that $t_{\max} = \infty$.

Step 2. Here, we show (3.2) and (3.3). First, by the arguments given in Step 1, $\sup_{t\ge 0} \|a(t,\cdot)\|_\infty \le \|\gamma\|_\infty + \|\beta\|_\infty \overline{S}$ and $I(t,x) \ge 0$ and solves (5.1). By the Harnack's inequality [16]*Theorem 2.5, there is a positive number c_* such that

$$\|I(t,\cdot)\|_\infty \le c_* \min_{x\in\overline{\Omega}} I(t,y), \quad t \ge 1. \tag{5.4}$$

Define

$$\overline{W}(t) = \max\Big\{\frac{N}{|\Omega|}, \Big\|\frac{\gamma}{\beta}\Big\|_\infty\Big\} + L_0 e^{-\sigma_0 \int_0^t \int_\Omega I(s,y)dyds} \quad t > 0,$$

and

$$\underline{W}(t) = \min\Big\{\frac{N}{|\Omega|}, \Big\|\frac{\gamma}{\beta}\Big\|_\infty\Big\} - L_1 e^{-\sigma_0 \int_0^t \int_\Omega I(s,y)dyds} \quad t > 0,$$

where $\sigma_0 = \dfrac{\beta_{\min}}{c_*|\Omega|} > 0$, $L_0 = \|S(1,\cdot)\|_\infty e^{\sigma_0 \int_0^1 \int_\Omega I(s,y)dyds}$, and L_1 is chosen so that $\underline{W}(1) = 0$.

By direct computation and using (5.4), we have that

$$\begin{aligned}
&\partial_t \overline{W} - d_S \Delta \overline{W} - \beta(x)\Big(\frac{\gamma(x)}{\beta(x)} - \overline{W}\Big) I(t,x) \\
=& -\sigma_0 L_0 e^{-\sigma_0 \int_0^t \int_\Omega I(s,y)dyds} \int_\Omega I(t,y)dy + \beta\Big(\max\Big\{\frac{N}{|\Omega|}, \Big\|\frac{\gamma}{\beta}\Big\|_\infty\Big\} - \frac{\gamma}{\beta} + L_0 e^{-\sigma_0 \int_0^t \int_\Omega I(s,y)dyds}\Big) I(t,x) \\
\ge& L_0\Big(\beta(x) I(t,x) - \sigma_0 \int_\Omega I(t,y)dy\Big) e^{-\sigma_0 \int_0^t \int_\Omega I(s,y)dyds} \\
\ge& L_0\Big(\beta(x) I(t,x) - \sigma_0 \int_\Omega I(t,y)dy\Big) e^{-\sigma_0 \int_0^t \int_\Omega I(s,y)dyds}
\end{aligned}$$

$$\geq L_0\Big(\beta_{min} - \sigma_0|\Omega|c_*\Big)e^{-\sigma_0\int_0^t\int_\Omega I(s,y)dyds}I(t,x) = 0$$

Thus, since $\overline{W}(1) \geq S(1,\cdot)$ and $\partial_{\vec{n}}\overline{W} = 0$ on $\partial\Omega$ then, by the comparison principle for parabolic equations, we have

$$S(t,\cdot) \leq \overline{W}(t), \quad t \geq 1. \tag{5.5}$$

Next, by direct computation and using (5.4) again, we have

$$\begin{aligned}
&\partial_t\underline{W} - d_S\Delta\underline{W} - \beta(x)\Big(\frac{\gamma(x)}{\beta(x)} - \underline{W}\Big)I(t,x)\\
=&\sigma_0 L_1 e^{-\sigma_0\int_0^t\int_\Omega I(s,y)dyds}\int_\Omega I(t,y)dy + \beta\Big(\min\Big\{\frac{N}{|\Omega|},\Big\|\frac{\gamma}{\beta}\Big\|_\infty\Big\} - \frac{\gamma}{\beta} - L_1e^{-\sigma_0\int_0^t\int_\Omega I(s,y)dyds}\Big)I(t,x)\\
\leq& L_1\Big(\sigma_0\int_\Omega I(t,y)dy - \beta(x)I(t,x)\Big)e^{-\sigma_0\int_0^t\int_\Omega I(s,y)dyds}\\
\leq& L_1\Big(\sigma_0|\Omega|I(t,x) - \beta(x)I(t,x)\Big)e^{-\sigma_1\int_0^t\int_\Omega I(s,y)dyds}\\
\leq& L_1\Big(\sigma_0|\Omega|c_* - \beta_{\min}\Big)e^{-\sigma_0\int_0^t\int_\Omega I(s,y)dyds}I(t,x) = 0
\end{aligned}$$

Thus, since $\underline{W}(1) = 0 \leq S(1,\cdot)$ and $\partial_{\vec{n}}\underline{W} = 0$ on $\partial\Omega$, then by the comparison principle for parabolic equations, we have

$$S(t,\cdot) \geq \underline{W}(t), \quad t \geq 1. \tag{5.6}$$

From this point, we distinguish two cases.

Case 1. Here, we suppose that $\displaystyle\int_0^\infty\int_\Omega I(t,x)dxdt = +\infty$. In this case, letting $t \to +\infty$ in both (5.5) and (5.6), we obtain that (3.2) and (3.3) hold.

Case 2. Here, we suppose that $\displaystyle\int_0^\infty\int_\Omega I(t,x)dxdt < +\infty$. Thus $\displaystyle\int_\Omega I(t,\cdot)dx \to 0$ as $t \to +\infty$. It then follows from the Harnack's inequality (5.4) that $\|I(t,\cdot)\|_\infty \to +\infty$ as $t \to +\infty$. Note also that $\displaystyle\int_\Omega S = N - \int_\Omega I \to N$ as $t \to +\infty$. We can now apply a perturbation argument to the first equation of (1.5) to derive that $\Big\|S(t,\cdot) - \dfrac{N}{|\Omega|}\Big\|_\infty \to 0$ as $t \to +\infty$, which implies that (3.2) and (3.3) hold.

Step 3. Here, we show that (3.1) holds. Due to (3.2) and (3.3), there is $t_1 > 0$ such that

$$\sup_{t\geq t_1}\|S(t,\cdot)\|_\infty \leq 2\max\Big\{\frac{N}{|\Omega|},\Big\|\frac{\gamma}{\beta}\Big\|_\infty\Big\}.$$

Therefore, we can apply the Harnack's inequality to derive that (5.4) for $t \geq t_1 + 1$, where the constant c_* depends only on d_I and $\Big\{\frac{N}{|\Omega|},\Big\|\frac{\gamma}{\beta}\Big\|_\infty\Big\}$. Therefore,

$$\|I(t,\cdot)\|_\infty \leq \frac{c_*}{|\Omega|}\int_\Omega I(t,x)dx \leq \frac{c_*N}{|\Omega|} := M^*, \quad t \geq t_1 + 1.$$

Hence, (3.1) holds.

□

5.1.2 Proof of Theorem 3.2.1

Proof of Theorem 3.2.1. Thanks to Lemma 2.3.1, studying the existence of endemic equilibrium solutions of (1.5) is equivalent to studying the existence of positive solutions of (2.13) satisfying (2.12). Hence, we introduce the function

$$\mathcal{N}_{d_I,d_S}(l) = \mathcal{N}_{d_I}(l) + \frac{l d_S}{l^*|\Omega|}\int_\Omega u^l, \quad l \geq l^* \tag{5.7}$$

where u^l is the unique positive solution of (2.17), $\mathcal{N}_{d_I}$ is introduced in (2.18), and l^* is introduced as in (2.16). By Lemma 2.4.2, the function $\mathcal{N}_{d_I,d_S}$ is of class C^1. Furthermore, by (2.20)-(2.21), it holds that

$$\mathcal{N}_{d_I,d_S}(l^*) = \lim_{l\to l^*}\mathcal{N}_{d_I,d_S}(l) = |\Omega|l^* = 1 \quad \text{and} \quad \lim_{l\to+\infty}\mathcal{N}_{d_I,d_S}(l) = +\infty. \tag{5.8}$$

By Lemma 2.3.1, (1.5) has an endemic equilibrium solution if and only if there is some $l > l^*$ such that $\mathcal{R}_0 = \mathcal{N}_{d_I,d_S}(l)$. Now, observe that

$$\frac{d\mathcal{N}_{d_I,d_I}(l)}{dl} = \frac{1}{l^*|\Omega|}\Big(|\Omega| + (d_S - d_I)\int_\Omega (lv^l + u^l)\Big), \quad l > l^*, \tag{5.9}$$

where v^l is the unique positive solution of (2.24). Next, by (2.20) and (2.26), we have that

$$\lim_{l\to+\infty} \frac{d\mathcal{N}_{d_I,d_S}(l)}{dl} = \frac{1}{l^*|\Omega|}\Big(|\Omega| + (d_S - d_I)\int_\Omega \frac{1}{d_I}\Big) = \frac{d_S}{d_I l^*} > 0.$$

Therefore, the quantity l_{d_I,d_S} defined by

$$l_{d_I,d_S} := \inf\{l > l^* \mid \frac{d\mathcal{N}_{d_I,d_S}(\tau)}{dl} > 0 \text{ for all } \tau > l\}$$

is well-defined. Define

$$\mathcal{R}_* := \max\{\mathcal{N}_{d_I,d_S}(l) \mid l \in [l^*, l_{d_I,d_S}]\}.$$

If $d_S \geq d_I$, then $\mathcal{N}_{d_I,d_S}$ is strictly increasing by (5.9), which implies that $l_{d_I,d_S} = l^*$, and hence $\mathcal{R}_* = \mathcal{N}_{d_I,d_S}(l^*) = 1$.

If $\frac{\gamma}{\beta} = c$ is constant, then $u^l = \frac{1}{d_I}(1 - \frac{c}{l})$ for all $l > l^*$, which yields

$$\frac{d\mathcal{N}_{d_I,d_S}(l)}{dl} = \frac{1}{l^*|\Omega|}\Big(|\Omega| + (d_S - d_I)|\Omega|\Big(l\frac{c}{d_I l^2} + \frac{1}{d_I}(1 - \frac{c}{l})\Big)\Big) = \frac{d_S}{d_I l^*} > 0, \quad l > l^*.$$

Thus, $\mathcal{N}_{d_I,d_S}(l)$ is strictly increasing and $l_{d_I,d_S} = l^*$. So, $\mathcal{R}_* = 1$ if $\frac{\gamma}{\beta}$ is constant.

(i) Suppose that $\mathcal{R}_0 > 1$. By (5.8) and the Intermediate Value Theorem, there is some $l_N > l^*$ such that $\mathcal{N}_{d_I,d_S}(l_N) = \mathcal{R}_0$. Thus the following set

$$\mathcal{S}_0 := \{l > l^* \mid \mathcal{N}_{d_I,d_S}(l) = \mathcal{R}_0\}$$

is not empty. We have that the set $\mathcal{S}_0$ contains exactly one element if $\mathcal{R}_0 > \mathcal{R}_*$, in which case (1.5) has unique endemic equilibrium solution by Lemma 2.3.1. Note that the function $\mathcal{N}_{d_I,d_S}$ is analytic on $(l^*, +\infty)$ since u^l is by the Implicit Function Theorem. Note also from (5.8) that the function $\mathcal{N}_{d_I,d_S}$ is not constant. Therefore, the set $\mathcal{S}_0$ has only finitely many elements. This shows that (1.5) has a finite number of endemic equilibrium solutions. If $\mathcal{S}_0$ contains $m \geq 2$ elements, say

$l_1 < l_2 < \cdots < l_m$, then by Lemma 2.3.1, the functions (S_i, I_i), $i = 1, \cdots, m$, defined by

$$S_i := l_i(1 - d_I u^{l_i}) \quad \text{and} \quad I_i := \frac{l_i d_S}{l^*|\Omega|} u^{l_i}$$

are the endemic equilibrium solutions of (1.5). Because of the monotonicity of u^l in l, we have that $I_i < I_{i+1}$, for $i = 1, \cdots, m-1$.

Finally, suppose, in addition, that either $\frac{\gamma}{\beta}$ is constant or $d_S \geq d_I$. Let (S^*, I^*) be the unique endemic equilibrium solution of (1.5). Linearizing (1.5) at (S^*, I^*), we obtain the linear parabolic system

$$\begin{cases} \partial_t S = d_S \Delta S - \beta I^* S + (\gamma - \beta S^*) I, & x \in \Omega,\ t > 0, \\ \partial_t I = d_I \Delta I + (\beta S^* - \gamma) I + \beta I^* S, & x \in \Omega,\ t > 0, \\ \partial_{\vec{n}} S = \partial_{\vec{n}} I = 0, & x \in \partial\Omega,\ t > 0, \\ \int_\Omega (S + I) = 0, & t > 0. \end{cases} \tag{5.10}$$

The eigenvalue problem associated with (5.10) is

$$\begin{cases} d_S \Delta \varphi_1 - \beta I^* \varphi_1 + (\gamma - \beta S^*) \varphi_2 + \lambda \varphi_1 = 0, & x \in \Omega, \\ d_I \Delta \varphi_2 + (\beta S^* - \gamma) \varphi_2 + \beta I^* \varphi_1 + \lambda \varphi_2 = 0, & x \in \Omega, \\ \partial_{\vec{n}} S = \partial_{\vec{n}} I = 0, & x \in \partial\Omega, \\ \int_\Omega (\varphi_1 + \varphi_2) = 0. \end{cases} \tag{5.11}$$

If $\frac{\gamma}{\beta}$ is constant, then (S^*, I^*) is spatially homogeneous and $\gamma - \beta S^* = 0$. In this case, (5.11) becomes

$$\begin{cases} d_S \Delta \varphi_1 - \beta I^* \varphi_1 + \lambda \varphi_1 = 0, & x \in \Omega, \\ d_I \Delta \varphi_2 + \beta I^* \varphi_1 + \lambda \varphi_2 = 0, & x \in \Omega, \\ \partial_{\vec{n}} S = \partial_{\vec{n}} I = 0, & x \in \partial\Omega, \\ \int_\Omega (\varphi_1 + \varphi_2) = 0. \end{cases} \tag{5.12}$$

If $\varphi_1 \neq 0$ then, by the first equation of (5.12), λ is real and $\lambda \geq \lambda(d_S, -\beta I^*) \geq (\beta I^*)_{\min} > 0$. If $\varphi_1 \equiv 0$ then, by the second equation of (5.12), we also have that λ is a real number. Moreover, $\lambda \geq \lambda(d_I, 0) = 0$ with equality if and only if φ_2 is a constant real number. In the latter case, it follows from the last equation of (5.12) that $\varphi_2 \equiv 0$, which is impossible. Therefore, if $\varphi_1 \equiv 0$, we must have that $\lambda > 0$. This shows that any eigenvalue of (5.12) is real and positive, so that (S^*, I^*) is linearly stable.

Now, suppose that $d_S \geq d_I$. We show that $\lambda = 0$ is not an eigenvalue of (5.11). Let λ be an eigenvalue of (5.11) and (φ_1, φ_2) be an associated eigenfunction. If $\varphi_2 \equiv 0$, then $\varphi_1 \neq 0$ and it follows from the first equation of (5.11) that λ is real with $\lambda \geq \lambda(d_S, -\beta I^*) \geq (\beta I^*)_{\min} > 0$. So we may suppose that $\varphi_2 \neq 0$. In this case, we proceed by contradiction to show that $\lambda \neq 0$. So, suppose to the contrary that $\lambda = 0$. Adding up the first two equations of (5.11) gives

$$0 = \Delta(d_S\varphi_1 + d_I\varphi_2), \quad x \in \Omega.$$

This, along with the boundary conditions, implies that $d_S\varphi_1 + d_I\varphi_2 = \kappa$ for some complex number $\kappa \in \mathbb{C}$. Therefore, by (5.11), we have that

$$\begin{cases} d_I\Delta\varphi_2 + \big((\beta S^* - \gamma) - \dfrac{d_I}{d_S}\beta I^*\big)\varphi_2 + \dfrac{\beta I^*}{d_I}\kappa = 0, & x \in \Omega, \\ \partial_{\vec{n}}\varphi_2 = 0, & x \in \partial\Omega, \\ \kappa = \Big(\dfrac{d_I}{d_S} - 1\Big)\dfrac{1}{|\Omega|}\displaystyle\int_\Omega \varphi_2 \end{cases} \tag{5.13}$$

Note that, since the functions $\big((\beta S^* - \gamma) - \frac{d_I}{d_S}\beta I^*\big)$ and $\frac{\beta I^*}{d_I}$ are real-valued and $\Big(\frac{d_I}{d_S} - 1\Big)\frac{1}{|\Omega|}$ is a real number, then then both the real part vector $(\mathrm{Re}(\varphi_2), \mathrm{Re}(\kappa))$ and the imaginary part $(\mathrm{Im}(\varphi_2), \mathrm{Im}(\kappa))$ satisfy (5.13). Therefore, we may suppose that φ_2 is a real-valued function and that κ is a real number. Next, since (S^*, I^*) is an endemic equilibrium solution of (1.5), then $\lambda(d_I, \beta S^* - \gamma) = 0$. As a result, by Lemma 2.4.1, we have that

$$\lambda(d_I, \beta S^* - \gamma - \frac{d_I}{d_I}\beta I^*) > \lambda(d_I, \beta S^* - \gamma) = 0.$$

Therefore, the linear operator $u \mapsto \big((\beta S^* - \gamma) - \frac{d_I}{d_S}\beta I^*)u - d_I\Delta u$, subject to the homogeneous Neumann boundary conditions, is invertible. Furthermore, by the strong maximum principle for elliptic equations, its inverse function is strongly positive. Therefore, by the first equation of (5.13), we have that φ_2 has the same sign as κ. This, along with the last equation of (5.13) and the fact that $\frac{d_I}{d_S} \leq 1$, implies that $\kappa = 0$, which in turn implies that $\varphi_2 \equiv 0$. So, we obtain a contradiction. Therefore, $\lambda = 0$ is not an eigenvalue of (5.11), and hence (S^*, I^*) is not degenerate.

(ii) Since for every $l > l^*$, $l(1 - d_I u^l) \geq (\gamma/\beta)_{\min}$ by the maximum principle for elliptic equations, then

$$\mathcal{N}_{d_I,d_S}(l) \geq \frac{|\Omega|(\gamma/\beta)_{\min}}{l^*|\Omega|} + l d_S \int_\Omega u^l = (\gamma/\beta)_{\min}\mathcal{R}_1 + l d_S \int_\Omega u^l > (\gamma/\beta)_{\min}, \quad l > l^*.$$

Hence, since (1.5) has an endemic equilibrium solution if and only if there is some $l > l^*$ such that $\mathcal{N}_{d_I,d_S}(l) = \mathcal{R}_0$, we conclude that (1.5) has no endemic equilibrium solution if $\mathcal{R}_0 \leq (\gamma/\beta)_{\min}\mathcal{R}_1$.

Finally, observe that

$$\begin{aligned}\mathcal{N}_{d_I,d_S}(l) =& \frac{1}{l^*|\Omega|}\Big(l|\Omega| + (l(d_S - d_I)_+ - l(d_S - d_I)_-)\int_\Omega u^l\Big) \\ >& \frac{l}{l^*}\Big(1 - (\frac{d_S}{d_I} - 1)_-\Big) = \frac{l}{l^*}\min\Big\{1, \frac{d_S}{d_I}\Big\} \geq \min\Big\{1, \frac{d_S}{d_I}\Big\}.\end{aligned}$$

Thus, (1.5) has no endemic equilibrium solution if $\mathcal{R}_0 \leq \min\Big\{1, \frac{d_S}{d_I}\Big\}$.

□

5.1.3 Proof of Theorem 3.3.1

Proof of Theorem 3.3.1. (i) Let d_I be a fixed positive number and suppose that $\mathcal{R}_0(N, d_I) > 1$ so that (3.1) has an endemic equilibrium solution $(S(\cdot, d_S), I(\cdot, d_S))$ for every $d_S > 0$. Suppose also that $N < \int_\Omega \frac{\gamma}{\beta}$. Let $(\kappa, \tilde{S}(\cdot), \tilde{I}(\cdot))$ be given by (2.9) and (2.10). We show that

$$\eta_0 := \liminf_{d_S \to 0} \int_\Omega (1 - d_I \tilde{I}(\cdot, d_S)) > 0. \tag{5.14}$$

Suppose that (5.14) were false. Then, there exists a sequence of positive numbers $\{d_{Sm}\}_{m\geq 1}$ converging to 0 such that

$$\lim_{m\to+\infty}\int_\Omega (1-d_I\tilde{I}(\cdot,d_{Sm}))=0. \tag{5.15}$$

Hence, equation (5.49) combined with (2.12) and the fact that $\displaystyle\int_\Omega \tilde{I}<\frac{1}{d_I}$ imply

$$\lim_{m\to+\infty}\frac{\kappa(d_{Sm})}{d_{Sm}}=\lim_{m\to+\infty}\frac{N}{\int_\Omega(1-d_I\tilde{I}(\cdot,d_{Sm}))+d_{Sm}\int_\Omega\tilde{I}(\cdot,d_{Sm})}=+\infty. \tag{5.16}$$

Now, by (2.13), $\tilde{I}(\cdot,d_{S_m})=u_{d_I,\frac{\kappa d_{S_m}}{d_{S_m}}}$ for all $m\geq 1$. So, we can conclude from (2.22) and (5.16) that

$$\lim_{m\to+\infty}\frac{\kappa(d_{Sm})}{d_{Sm}}(1-d_I\tilde{I}(\cdot,d_{Sm}))=\frac{\gamma}{\beta}$$

uniformly in $\overline{\Omega}$. Therefore, since $S(\cdot,d_{Sm})=\dfrac{\kappa(d_{Sm})}{d_{Sm}}(1-d_I\tilde{I}(\cdot,d_{Sm}))$ and $N=\displaystyle\int_\Omega(S+I)\geq\int_\Omega S$, we have that

$$N\geq\lim_{m\to+\infty}\int_\Omega S(\cdot,d_{Sm})=\int_\Omega\frac{\gamma}{\beta},$$

which contradicts our assumption that $N<\displaystyle\int_\Omega\frac{\gamma}{\beta}$. Therefore, (5.14) holds and so

$$\limsup_{d_S\to 0^+}\frac{\kappa(d_S)}{d_S}=\limsup_{d_S\to 0^+}\frac{N}{\int_\Omega(1-d_I\tilde{I})+d_S\int_\Omega\tilde{I}}\leq\frac{N}{\eta_0}.$$

Therefore, there is a $C_1>0$ such that

$$\frac{\kappa(d_S)}{d_S}\leq C_1,\quad 0<d_S<1. \tag{5.17}$$

Next, we show that

$$\eta_1:=\liminf_{d_S\to 0}\min_{x\in\overline{\Omega}}\tilde{I}(x,d_S)>0. \tag{5.18}$$

Suppose that (5.18) were false. Observe from (5.17) and the fact that $0<d_I\tilde{I}(\cdot,d_S)<1$ that

$$\left\|\frac{\beta\kappa(d_S)}{d_S}(1-d_I\tilde{I}(\cdot,d_S))-\gamma\right\|_\infty\leq C_1\|\beta\|_\infty+\|\gamma\|_\infty,\quad 0<d_S<1. \tag{5.19}$$

Then, by (5.19) and the Harnack's inequality for elliptic equations, there exists a sequence of positive numbers $\{d_{Sm}\}_{m\geq 1}$ converging to 0 such that $\left\|\tilde{I}(\cdot, d_{Sm})\right\|_\infty \to 0$ as $m \to +\infty$. Therefore,

$$\lim_{m\to+\infty} \frac{\kappa(d_{Sm})}{d_{Sm}} = \lim_{m\to+\infty} \frac{N}{\int_\Omega (1 - d_I\tilde{I}(\cdot, d_{Sm})) + d_{Sm}\tilde{I}(\cdot, d_{Sm})} = \frac{N}{|\Omega|}.$$

Therefore, if we define $\hat{I} = \dfrac{\tilde{I}}{\left\|\tilde{I}\right\|_\infty}$, then it follows from the regularity theory for elliptic equations that $\hat{I}(\cdot, d_{Sm}) \to \hat{I}^*$ (up to a subsequence) as $m \to +\infty$ for some function $\hat{I}^* \in C^2(\Omega)$ satisfying $\hat{I}^* \geq 0$, $\left\|\hat{I}^*\right\|_\infty = 1$, and

$$\begin{cases} d_I\Delta\hat{I}^* + \left(\dfrac{N\beta}{|\Omega|} - \gamma\right)\hat{I}^* = 0, & x \in \Omega, \\ \partial_{\vec{n}}\hat{I}^* = 0, & x \in \partial\Omega, \end{cases}$$

which, by (2.2), implies that $\mathcal{R}_0 = \dfrac{N}{|\Omega|}\mathcal{R}_1 = 1$. This contradicts our assumption that $\mathcal{R}_0 = \mathcal{R}_0(N, d_I) > 1$. Therefore, (5.18) holds.

Now, from (5.17), we have that

$$\|I(\cdot, d_S)\|_\infty = \kappa(d_S)\left\|\tilde{I}(\cdot, d_S)\right\|_\infty \leq C_1 d_S, \quad 0 < d_S < 1. \tag{5.20}$$

On the other hand, applying the maximum principle to the equation for S, we get

$$\min_{x\in\overline{\Omega}} \frac{\gamma}{\beta} \leq S(\cdot, d_S) \leq \max_{x\in\overline{\Omega}} \frac{\gamma}{\beta}. \tag{5.21}$$

Hence,

$$\frac{\kappa(d_S)}{d_S} = S + \frac{d_I}{d_S}I \geq \min_{x\in\overline{\Omega}} \frac{\gamma}{\beta}, \quad d_S > 0. \tag{5.22}$$

Then, (5.22) combined with (5.18) yield that

$$\liminf_{d_S\to 0} \min_{x\in\overline{\Omega}} \frac{I(\cdot, d_S)}{d_S} = \liminf_{d_S\to 0} \frac{\kappa(d_S)}{d_S} \min_{x\in\overline{\Omega}} \tilde{I}(\cdot, d_S) \geq \eta_1 \min_{x\in\overline{\Omega}} \frac{\gamma}{\beta} > 0. \tag{5.23}$$

Thus, (3.4) follows from (5.20) and (5.23).

Finally, by (5.14), (5.18), (5.19), and the fact that $0 < d_I\tilde{I} < 1$, we can apply the regularity theory for elliptic equations to obtain that there is $\tilde{I}^* \in C^{2,\mu}(\Omega)$, $0 < \mu \ll 1$, satisfying $0 < \tilde{I}^* < \dfrac{1}{d_I}$ and (3.6) such that $\tilde{I}(\cdot, d_S) \to \tilde{I}^*$ as $d_S \to 0$ (up to a subsequence) in $C^2(\Omega)$. Therefore,

$$S = \frac{\kappa(d_S)}{d_S}(1 - d_I\tilde{I}) = \frac{N(1 - d_I\tilde{I})}{\int_\Omega(1 - d_I\tilde{I}) + d_S\int_\Omega \tilde{I}} \to S* := \frac{N(1 - d_I\tilde{I}^*)}{\int_\Omega(1 - d_I\tilde{I}^*)} \quad \text{as} \quad d_S \to 0,$$

uniformly on $C^2(\Omega)$ and so (3.5) holds.

(ii) Now, suppose that $N > \displaystyle\int_\Omega \frac{\gamma}{\beta}$. By Lemma 2.4.2, the elliptic equation (2.17) has a unique positive solution $0 < u_{d_I,l} < \dfrac{1}{d_I}$ for every $l > \dfrac{1}{\mathcal{R}_1}$. Moreover,

$$\lim_{l \to +\infty} u_{d_I,l} = \frac{1}{d_I} \quad \text{and} \quad \lim_{l \to +\infty} l(1 - d_I u_{d_I,l}) = \frac{\gamma}{\beta} \tag{5.24}$$

uniformly in $x \in \overline{\Omega}$. Define, for each $l > \dfrac{1}{\mathcal{R}_1}$,

$$d_{Sl} := \frac{N - l\int_\Omega(1 - d_I u_{d_I,l})}{l\int_\Omega u_{d_I,l}}. \tag{5.25}$$

We then have that, from (5.24) and the fact that $N > \displaystyle\int_\Omega \frac{\gamma}{\beta}$, there exists $l_0 \gg 1$ such that $d_{Sl} > 0$ for $l > l_0$. Take

$$S(\cdot, d_{Sl}) = l(1 - d_I u_{d_I,l}) \quad \text{and} \quad I(\cdot, d_{Sl}) = \left(N - \int_\Omega S(\cdot, d_{Sl})\right)\frac{u_{d_I,l}}{\int_\Omega u_{d_I,l}}, \quad l > l_0.$$

Then, it can be easily verified that $(S(\cdot, d_{Sl}), I(\cdot, d_{Sl}))$ is an endemic equilibrium solution of (3.1) for every $l > l_0$. Moreover, we have from (5.24) that $(S(\cdot, d_{Sl}), I(\cdot, d_{Sl})) \to \left(\dfrac{\gamma}{\beta}, \dfrac{1}{|\Omega|}\left(N - \displaystyle\int_\Omega \frac{\gamma}{\beta}\right)\right)$ as $l \to +\infty$, uniformly in $\overline{\Omega}$. Finally, using the regularity of the function $l \mapsto u_{d_I,l}$ and using (5.25), we have that the function $l \mapsto d_{Sl}$ is continuous and

$$\lim_{l \to +\infty} d_{Sl} = 0.$$

□

5.1.4 Proof of Theorem 3.3.2

Proof of Theorem 3.3.2. Suppose that $H^+ \neq \emptyset$.

(i) Since H^+ is nonempty, then by Theorem 3.2.1 (i) and Lemma 2.2.1, there exists a $0 < d_0 \ll 1$ such that (1.5) has an endemic equilibrium $(S(\cdot), I(\cdot))$ for every $d_S > 0$ and $0 < d_I \leq d_0$. Observe from Theorem 3.2.1 (i) that the endemic equilibrium $(S(\cdot), I(\cdot))$ of (1.5) is unique whenever $d_S \geq d_I$, in particular for $d_I < \min\{d_0, d_S\}$. Let $(\kappa, \tilde{S}(\cdot), \tilde{I}(\cdot))$ be given by (2.9) and (2.10). Observe that

$$\frac{\kappa}{d_S} = \frac{N}{\int_\Omega (1 - d_I\tilde{I}) + d_S\tilde{I}} = \frac{N}{|\Omega| + (d_S - d_I)\int_\Omega \tilde{I}} \leq \frac{N}{|\Omega|} \quad \text{for all} \quad 0 < d_I < \min\{d_0, d_S\}$$

which combined with (5.22) gives

$$\min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)} \leq \frac{\kappa}{d_S} \leq \frac{N}{|\Omega|} \quad \text{for all} \quad 0 < d_I < \min\{d_0, d_S\}. \tag{5.26}$$

The rest of the proof will be divided into three steps.

Step 1. In this step, we will show that (3.8) holds. We begin by claiming that

$$\lim_{\max\{d_I, \frac{d_I}{d_S}\}\to 0} \frac{\kappa}{d_S} = \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}. \tag{5.27}$$

Suppose, for the sake of contradiction, that (5.27) were false. Then, we can find sequences of positive numbers $\{d_{I_m}\}_{m\geq 1}$ and $\{d_{S_m}\}_{m\geq 1}$ satisfying $\max\{d_{I_m}, \frac{d_{I_m}}{d_{S_m}}\} \to 0$ as $m \to +\infty$ such that

$$\frac{N}{|\Omega|} \geq \eta_2 := \lim_{m\to+\infty} \frac{\kappa}{d_{S_m}} > \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}. \tag{5.28}$$

Observe that the function $\tilde{u}(\cdot) = d_I\tilde{I}(\cdot)$ satisfies

$$\begin{cases} d_I\Delta\tilde{u} + \Big(\frac{\beta\kappa}{d_S}(1-\tilde{u}) - \gamma\Big)\tilde{u} = 0, & x \in \Omega, \\ \partial_{\vec{n}}\tilde{u} = 0, & x \in \partial\Omega, \end{cases} \tag{5.29}$$

for every $d_I > 0$. Since $\max\{d_{I_m}, \frac{d_{I_m}}{d_{S_m}}\} \to 0$ and $\frac{\kappa}{d_{S_m}} \to \eta_2$, it follows from the singular perturbation theory (see [5]) that

$$\lim_{m\to+\infty}\tilde{u}=\Big(1-\frac{\gamma}{\eta_2\beta}\Big)_+ \tag{5.30}$$

uniformly on $\overline{\Omega}$. Additionally, we have from (5.26) that

$$\int_\Omega \tilde{u}=\frac{d_I}{\kappa}\int_\Omega I\le d_I\frac{N}{\kappa}\le\frac{d_I}{d_S}\frac{N}{\min_{x\in\overline{\Omega}}\frac{\gamma(x)}{\beta(x)}}\to 0\quad\text{as }\frac{d_I}{d_S}\to 0.$$

Therefore, we can conclude from (5.30) that

$$\Big(1-\frac{\gamma(x)}{\eta_2\beta(x)}\Big)_+=0,\quad x\in\Omega,$$

which yields that

$$\eta_2\le\min_{x\in\overline{\Omega}}\frac{\gamma(x)}{\beta(x)}.$$

This contradicts (5.28). Thus, we have that (5.27) must hold. From (5.21), we get

$$\min_{x\in\overline{\Omega}}\frac{\gamma(x)}{\beta(x)}\le\min_{x\in\overline{\Omega}}S(x)\le\max_{x\in\overline{\Omega}}S(x)\le\frac{\kappa}{d_S}\to\min_{x\in\overline{\Omega}}\frac{\gamma(x)}{\beta(x)}\quad\text{ as }\max\{d_I,\frac{d_I}{d_S}\}\to 0.$$

Therefore,

$$S\to\min_{x\in\overline{\Omega}}\frac{\gamma(x)}{\beta(x)}\text{ as }\max\{d_I,\frac{d_I}{d_S}\}\to 0\text{ (uniformly on }\overline{\Omega}) \tag{5.31}$$

and

$$\int_\Omega I=N-\int_\Omega S\to N-|\Omega|\min_{x\in\overline{\Omega}}\frac{\gamma(x)}{\beta(x)}\quad\text{as}\quad\max\{d_I,\frac{d_I}{d_S}\}\to 0. \tag{5.32}$$

So, (3.8) holds.

Step 2. Here, we will show that (3.10) holds. Let K be a compact subset of $\Omega\setminus\Omega^*$. Let O_1 and O_2 be two open sets such that $K\subset O_1\subset\overline{O}_1\subset O_2\subset\overline{O}_2\subset\Omega\setminus\Omega^*$. Using (5.31), the definition of the set Ω^*, and the fact that $\overline{O}_2\cap\Omega^*=\emptyset$, there are $0<d_*\ll 1$ and $\eta_*>0$ such that

$$\eta_*\le\frac{\gamma(x)}{\beta(x)}-S(x)\text{ for every }x\in\overline{O}_2\text{ whenever }\max\{d_I,\frac{d_I}{d_S}\}<d_*. \tag{5.33}$$

As a result,

$$d_I\Delta I=(\gamma-S\beta)I>0,\quad x\in O_2,\ \max\{d_I,\frac{d_I}{d_S}\}<d_*.$$

This shows that, if $\max\{d_I,\frac{d_I}{d_S}\}<d_*$, then I is subharmonic on O_2. Hence, since $K\subset\subset O_1\subset\subset O_2$, there is a positive constant $c_*>0$ such that

$$\max_{x\in K}I(x)\le c_*\int_{\overline{O_1}}I(y)\ dy\quad\text{for all}\quad\max\{d_I,\frac{d_I}{d_S}\}<d_*.\tag{5.34}$$

Next, choose a smooth function φ satisfying $\|\varphi\|_{C^2(\overline{\Omega})}<+\infty$, $0\le\varphi\le 1$, $\varphi=1$ on $\overline{O}_1$, and $\varphi=0$ on $\Omega\setminus O_2$. Multiply the equation for I by φ and integrate by parts to obtain

$$\int_\Omega(\gamma-\beta S)I\varphi=d_I\int_\Omega I\Delta\varphi\le d_I\|\Delta\varphi\|_\infty\int_\Omega I\le d_IN\|\Delta\varphi\|_\infty.\tag{5.35}$$

Observe from (5.33) and (5.34) that, when $\max\{d_I,\frac{d_I}{d_S}\}<d_*$,

$$\int_\Omega(\gamma-\beta S)I\varphi=\int_{O_2}(\gamma-\beta S)I\varphi\ge\eta_*\min_x\beta(x)\int_{O_2}I\varphi\ge\eta_*\min_x\beta(x)\int_{\overline{O}_1}I\ge\frac{\eta_*}{c_*}\min_x\beta(x)\max_{x\in K}I(x),$$

Therefore, by (5.34), $\max_{x\in K}I(x)\le\frac{c_*d_IN\|\Delta\varphi\|_\infty}{\eta^*\min_x\beta(x)}\to 0$ as $\max\{d_I,\frac{d_I}{d_S}\}\to 0$.

Step 3. In this step, we show that there is a Radon probability measure on $\overline{\Omega}$ such that (3.11) and (3.12) hold. By (5.32), there is a Radon probability measure μ on $\overline{\Omega}$ and a sequence $\{(d_{I,m},d_{S,m})\}$ satisfying $\max\{d_{I,m},\frac{d_{I,m}}{d_{S,m}}\}\to 0$ as $m\to+\infty$ such that (3.11) holds in the sense that

$$\lim_{m\to+\infty}\int_{\overline{\Omega}}fI=\int_{\overline{\Omega}}f\ d\mu,\quad\text{for all}\quad f\in C(\overline{\Omega}).\tag{5.36}$$

Next, if we integrate the second equation of (2.1), we get

$$0=\int_{\Omega}(\beta S(\cdot,d_{S,m},d_{I,m})-\gamma)I(\cdot,d_{S,m},d_{I,m}),\quad m>1.$$

Letting $m \to +\infty$ in this equation and using (5.31) and (5.36), then we have that

$$0 = \int_{\overline{\Omega}} (\min_{y \in \overline{\Omega}} \frac{\gamma}{\beta} - \frac{\gamma}{\beta}) \beta \; d\mu,$$

which implies that $(\min_{y\in\overline{\Omega}} \frac{\gamma(y)}{\beta(y)} - \frac{\gamma(x)}{\beta(x)})\beta(x) = 0$ μ- almost everywhere since $(\min_{y\in\overline{\Omega}} \frac{\gamma(y)}{\beta(y)} - \frac{\gamma(x)}{\beta(x)})\beta(x) \leq 0$ for every $x \in \overline{\Omega}$. Therefore, $\mu(\overline{\Omega} \setminus \Omega^*) = 0$ since $(\min_{y\in\overline{\Omega}} \frac{\gamma(y)}{\beta(y)} - \frac{\gamma(x)}{\beta(x)})\beta(x) < 0$ for every $x \in \overline{\Omega} \setminus \Omega^*$.

(ii) Suppose $N < \int_{\Omega} \frac{\gamma}{\beta}$. Using the same arguments from (i), we know that there is a $0 < d_0 \ll 1$ such that (1.5) has an endemic equilibrium solution $(S(\cdot), I(\cdot))$ for every $d_S > 0$ and $0 < d_I \leq d_0$ since H^+ is not an empty set.

Next, we will show that (3.13) and (3.14) hold. We begin by first claiming that

$$\nu_0 := \liminf_{\max\{d_I, \frac{d_S}{d_I}\} \to 0} \int_{\Omega} (1 - d_I \tilde{I}) > 0. \tag{5.37}$$

Suppose (5.37) were false. Then, there is a sequence $\{(d_{I_m}, d_{S_m})\}_{m\geq 1}$ of positive pairs with $\max\{d_{I_m}, \frac{d_{S_m}}{d_{I_m}}\} \to 0$ such that

$$\lim_{m\to+\infty} \int_{\Omega} (1 - d_{I_m}\tilde{I}) = 0.$$

This, together with the fact that $\int_{\Omega} d_I \tilde{I} < 1$, implies that

$$\lim_{m\to+\infty} \frac{\kappa}{d_{S_m}} = \lim_{m\to+\infty} \frac{N}{\int_{\Omega}(1 - d_{I_m}\tilde{I}) + \frac{d_{S_m}}{d_{I_m}} \int_{\Omega} d_{I_m}\tilde{I}} = +\infty. \tag{5.38}$$

Recalling that $S = \frac{\kappa}{d_{S_m}}(1 - d_{I_m}\tilde{I})$, it follows from (5.21) that

$$\frac{d_{S_m}}{\kappa} \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)} \leq (1 - d_{I_m}\tilde{I}) \leq \frac{d_{S_m}}{\kappa} \max_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}, \quad m \geq 1,$$

which implies that

$$\lim_{m\to+\infty} d_{I_m}\tilde{I} \to 1 \quad \text{uniformly on } \; \overline{\Omega}, \tag{5.39}$$

since (5.38) holds. Observe also that

$$\begin{cases} d_{I_m}\dfrac{d_{S_m}}{\kappa}\Delta S + (\beta S - \gamma)(d_{I_m}\tilde{I}) = 0, & x \in \Omega, \\ \partial_{\vec{n}} S = 0, & x \in \partial\Omega, \end{cases} \tag{5.40}$$

for every $m \geq 1$. In view of (5.39) and the fact that $d_{I_m}\dfrac{d_{S_m}}{\kappa} \to 0$ as $m \to +\infty$, it follows from singular perturbation theory that

$$S \to \frac{\gamma}{\beta} \quad \text{as} \quad m \to +\infty$$

uniformly on $\overline{\Omega}$, which in turn yields that

$$N \geq \lim_{m\to+\infty} \int_\Omega S = \int_\Omega \frac{\gamma}{\beta}.$$

Consequently, we have a contradiction to our assumption that $N < \displaystyle\int_\Omega \frac{\gamma}{\beta}$. Therefore, we obtain that (5.37) must hold. Next, observe that

$$\limsup_{\max\{d_I, \frac{d_S}{d_I}\}\to 0^+} \frac{\kappa}{d_S} = \limsup_{\max\{d_I, \frac{d_S}{d_I}\}\to 0^+} \frac{N}{\int_\Omega (1 - d_I\tilde{I}(\cdot, d_I, d_S)) + d_S\int_\Omega \tilde{I}(\cdot, d_I, d_S)} \leq \frac{N}{\nu_0}.$$

This tells us that there exists a $C_1 > 0$ such that

$$\frac{\kappa}{d_S} \leq C_1 \quad \text{for all} \quad \max\{d_I, \frac{d_S}{d_I}\} < d_0. \tag{5.41}$$

Next, we claim that

$$\nu_1 := \liminf_{\max\{d_I, \frac{d_S}{d_I}\}\to 0} \|d_I\tilde{I}(x, d_I, d_S)\|_\infty > 0. \tag{5.42}$$

For sake of contradiction, suppose that (5.42) were false. Then, there exists a sequence of positive pairs (d_{I_m}, d_{S_m}) with $\max\{d_{I_m}, \dfrac{d_{S_m}}{d_{I_m}}\} \to 0$ as $m \to +\infty$ such that

$$\tilde{u}_m := d_{I_m}\tilde{I} \to 0 \quad \text{as } m \to +\infty. \tag{5.43}$$

Since $\frac{\kappa}{d_S} \geq \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}$, in view of (5.41) and the Bolzano-Weierstrass theorem, we may suppose, without loss of generality, that there is $\nu_2 \in [\min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}, C_1]$ such that

$$\lim_{m\to+\infty} \frac{\kappa}{d_{S_m}} = \nu_2.$$

Now, observe that the function $\tilde{u}_m = d_{I_m}\tilde{I}$ satisfies (5.29). Then, as in (5.30), we have that

$$\lim_{m\to+\infty} \tilde{u}_m = \Big(1 - \frac{\gamma}{\nu_2\beta}\Big)_+$$

which together with (5.43) yields that $\Big(1 - \frac{\gamma}{\nu_2\beta}\Big)_+ \equiv 0$, that is $\nu_2 \leq \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}$. So, we get $\nu_2 = \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}$ and, hence, we may use similar arguments that led to (5.31) and (5.32) to obtain that

$$S \to \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)} \quad \text{and} \quad \int_\Omega I \to N - |\Omega| \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)} \quad \text{as} \quad m \to +\infty. \tag{5.44}$$

But, by (5.41), we have

$$\|I\|_\infty \leq \frac{\kappa}{d_{I_m}} = \Big(\frac{d_{S_m}}{d_{I_m}}\Big)\frac{\kappa}{d_{S_m}} \leq C_1 \frac{d_{S_m}}{d_{I_m}} \to 0 \quad \text{as} \quad m \to +\infty,$$

which together with (5.44) yields that $\frac{N}{|\Omega|} = \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}$. Therefore, H^+ is an empty set, which is a contradiction. So, (5.42) must hold. Now, by (5.42), there is a $C_2 > 0$ such that

$$C_2 \leq \|d_I\tilde{I}\|_\infty \quad \text{for all} \quad \max\{d_I, \frac{d_S}{d_I}\} < d_0. \tag{5.45}$$

Inequality (5.41) yields

$$\|I\|_\infty \leq \frac{\kappa}{d_I} \leq C_1 \frac{d_S}{d_I} \quad \text{for all} \quad \max\{d_I, \frac{d_S}{d_I}\} < d_0,$$

while from (5.45), it holds that

$$\|I\|_\infty = \frac{\kappa}{d_I}\|d_I\tilde{I}\|_\infty \geq \Big[C_2 \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}\Big]\frac{d_S}{d_I} \quad \text{for all} \quad \max\{d_I, \frac{d_S}{d_I}\} < d_0.$$

Hence, (3.13) follows. Next, we show (3.14).

From the arguments given above, we can see that, up to a subsequence, $\frac{\kappa}{d_S} \to \nu \in \big[\min_x \frac{\gamma}{\beta}, C_1\big]$. Moreover, up to a further subsequence, $\tilde{u} = d\tilde{I} \to \Big(1 - \frac{\gamma}{\nu\beta}\Big)_+$ as $\max\{d_I, \frac{d_S}{d_I}\} \to 0$ uniformly on $\overline{\Omega}$. We have seen that $\Big\|\Big(1 - \frac{\gamma}{\nu\beta}\Big)_+\Big\|_\infty > 0$, that is $\nu > \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}$. Observe also that, up to a subsequence,

$$S = \frac{\kappa}{d_S}(1-\tilde{u}) \to \nu\Big(1 - \big(1 - \frac{\gamma}{\nu\beta}\big)_+\Big) \quad \text{as } \max\{d_I, \frac{d_S}{d_I}\} \to 0 \text{ uniformly on } \overline{\Omega}. \tag{5.46}$$

But, if it were the case that $\nu \geq \max_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}$ (i.e. $\big(1 - \frac{\gamma}{\nu\beta}\big)_+ = 1 - \frac{\gamma}{\nu\beta}$ on Ω), we would obtain that $S \to \frac{\gamma}{\beta}$ as $\max\{d_I, \frac{d_S}{d_I}\} \to 0$, up to a subsequence. This, combined with the fact that $\|I\|_\infty \to 0$ as $\max\{d_I, \frac{d_S}{d_I}\} \to 0$, yields

$$N = \int_\Omega \frac{\gamma}{\beta},$$

which is not possible since $N < \int_\Omega \frac{\gamma}{\beta}$. Therefore, it must also be the case that $\nu < \max_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}$. Finally, since $\|I\|_\infty \to 0$ as $\max\{d_I, \frac{d_S}{d_I}\} \to 0$ and (5.46) holds, then $N = \nu \int_\Omega \Big(1 - \big(1 - \frac{\gamma}{\nu\beta}\big)_+\Big) = \int_\Omega \min\{\nu, \frac{\gamma}{\beta}\}$. Hence, because the function $\big(\min_{x\in\overline{\Omega}} \frac{\gamma}{\beta}, \max_{x\in\overline{\Omega}} \frac{\gamma}{\beta}\big) \ni \nu \mapsto \int_\Omega \min\{\nu, \frac{\gamma}{\beta}\}$ is strictly decreasing, then ν is uniquely determined by $N = \int_\Omega \min\{\nu, \frac{\gamma}{\beta}\}$. This implies that the limit of S in (5.46) does not depend on the chosen subsequence. Finally, observe that

$$N = \int_\Omega \min\{\nu, \frac{\gamma}{\beta}\} \quad \text{and} \quad \min_{x\in\overline{\Omega}} \frac{\gamma}{\beta} < \nu < \max_{x\in\overline{\Omega}} \frac{\gamma}{\beta} \quad \text{imply} \quad N < \int_\Omega \nu = \nu|\Omega|.$$

(iii) Suppose that $\mathcal{R}_1 > 1$. By Lemma 2.2.1, there is $0 < d_0 \ll 1$ such that $\frac{1}{d_I} > \frac{1}{\mathcal{R}(d_I)}$ for every $0 < d_I \leq d_0$ since $\frac{\int_\Omega \beta}{\int_\Omega \gamma} \leq \mathcal{R}(d_I) \leq \|\frac{\beta}{\gamma}\|_\infty$. For every $0 < d_I < d_0$, with $l_I = \frac{1}{d_I}$, it follows from Lemma 2.4.2-(i), that (2.17) has a unique positive solution $0 < u_{d_I,l_I} < \frac{1}{d_I}$. By taking $w_I = l_I(1 - d_I u_{d_I,l_I})$, we know that w_I satisfies (2.30). Hence, by the maximum principle for elliptic equations, we get that

$$\min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)} \leq w_I \leq \max_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}, \quad 0 < d_I < d_0,$$

which is equivalent to saying that

$$d_I \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)} = \frac{1}{l_I} \min_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)} \le 1 - d_I u_{d_I,l_I} \le \frac{1}{l_I} \max_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)} = d_I \max_{x\in\overline{\Omega}} \frac{\gamma(x)}{\beta(x)}, \quad 0 < d_I < d_0.$$

Hence

$$\tilde{u}_I := d_I u_{d_I,l_I} \to 1 \quad \text{as } d_I \to 0 \tag{5.47}$$

uniformly on $\overline{\Omega}$. Next, multiplying (2.30) by d_I, we get

$$\begin{cases} 0 = \dfrac{d_I}{l_I}\Delta w_I + (\gamma - \beta w_I)\tilde{u}_I & x \in \Omega, \\ 0 = \partial_{\vec{n}} w_I & x \in \partial\Omega. \end{cases}$$

Therefore, since $\tilde{u}_I \to 1$ and $\dfrac{d_I}{l_I} = d_I^2 \to 0$ as $d_I \to 0$, we can make use of singular perturbation theory to conclude that $w_I \to \dfrac{\gamma}{\beta}$ uniformly on $\overline{\Omega}$ as $d_I \to 0$. This implies that

$$N - \int_\Omega w_I \to N - \int_\Omega \frac{\gamma}{\beta} > 0 \quad \text{as} \quad d_I \to 0.$$

So, there is $0 < d_1 \ll d_0$ such that $N - \displaystyle\int_\Omega w_I > 0$ for every $0 < d_I < d_1$. Now, define

$$d_{S,I} = \frac{N - \int_\Omega w_I}{l_I \int_\Omega u_{d_I,l_I}}, \quad 0 < d_I < d_1.$$

We have that $d_{S,I} > 0$ for every $0 < d_I < d_1$ and

$$\frac{d_{S,I}}{d_I} = \frac{N - \int_\Omega w_I}{l_I \int_\Omega d_I u_{d_I,l_I}} = d_I \frac{(N - \int_\Omega w_I)}{\int_\Omega \tilde{u}_I} \to 0 \quad \text{as } d_I \to 0.$$

Moreover, taking

$$S(\cdot, d_I, d_{S,I}) = w_I \quad \text{and} \quad I(\cdot, d_I, d_{S,I}) = \Big(N - \int_\Omega w_I\Big)\frac{\tilde{u}_I}{\int_\Omega \tilde{u}_I}, \quad 0 < d_I < d_1,$$

we have that $(S(\cdot, d_I, d_{S,I}), I(\cdot, d_I, d_{S,I}))$ is an endemic equilibrium solution of (1.5) with $d_S = d_{S,I}$.

Furthermore, it follows from above that

$$S(\cdot, d_I, d_{S,I}) \to \frac{\gamma}{\beta} \quad \text{and} \quad \int_\Omega I(\cdot, d_I, d_{S,I}) \to N - \int_\Omega \frac{\gamma}{\beta} \quad \text{as} \quad d_I \to 0.$$

This completes the proof of (3.16). ☐

5.1.5 Proof of Theorem 3.3.3

Proof of Theorem 3.3.3. Suppose that H^+ is nonempty and $N < \int_\Omega \frac{\gamma}{\beta}$. By Lemma 2.3.1 and Theorem 3.2.1, there is $0 < d_0 \ll 1$ such that (1.5) has an endemic equilibrium solution $(S(\cdot), I(\cdot))$ for every $d_S > 0$ and $0 < d_I < d_0$. By Theorem 3.3.1-(i), up to a subsequence, $S(\cdot) \to S^*(\cdot, d_I)$ uniformly on $\overline{\Omega}$ as $\frac{d_S}{d_I} \to 0$ for every $0 < d_I < d_0$. But by Theorem 3.3.2-(ii), we know that $S \to S_{\nu^*}$ uniformly on $\overline{\Omega}$ as $\max\{d_I, \frac{d_S}{d_I}\} \to 0$. Hence, we must have that $S(\cdot, d_I) \to S_{\nu^*}$ uniformly on $\overline{\Omega}$ as $d_I \to 0$. ☐

5.1.6 Proof of Theorem 3.4.1

Proof of Theorem 3.4.1. Suppose that $d_S = 0$ and $d_I > 0$.

(i) Suppose that $N \le \int_\Omega \frac{\gamma}{\beta}$. Let (S, I) be an equilibrium solution of (3.1). Then, (S, I) satisfies

$$\begin{cases} (\gamma - \beta S)I = 0, & x \in \Omega, \\ d_I \Delta I = 0, & x \in \Omega, \\ \partial_{\vec{n}} I = 0, & x \in \partial\Omega \\ N = \int_\Omega (S + I). \end{cases} \tag{5.48}$$

Thus, we see that $I \equiv c$ for some nonnegative constant c. If c is positive, then from the first equation in (5.48), it follows that $S = \frac{\gamma}{\beta}$. From the last equation of (5.48), we get

$$c|\Omega| = N - \int_\Omega \frac{\gamma}{\beta} \le 0.$$

Thus, we get $c = 0$ which means there is no endemic equilibrium solution to (3.1) in this case.

Clearly, if we have some nonnegative function $S \in C(\overline{\Omega})$ that satisfies $\int_\Omega S = N$, then $(S, 0)$ is a disease-free equilibrium solution of (3.1).

Next, suppose that $N < \int_\Omega \frac{\gamma}{\beta}$. Suppose also that $(S(t,x), I(t,x))$ is a classical solution of (3.1) with positive initial data. Define

$$U(t,x) = \frac{\beta(x)}{\gamma(x)} S(t,x), \quad t \geq 0, x \in \Omega,$$

and the sets

$$\Omega_- := \{x \in \overline{\Omega} \mid U(0,x) < 1\}$$
$$\Omega_0 := \{x \in \overline{\Omega} \mid U(0,x) = 1\}$$
$$\Omega_+ := \{x \in \overline{\Omega} \mid U(0,x) > 1\}.$$

Note that $(U(t,x), I(t,x))$ satisfies

$$\begin{cases} \partial_t U = \beta(1-U)I, & x \in \Omega, \quad t > 0, \\ \partial_t I = d_I \Delta I + \gamma(U-1)I, & x \in \Omega, \quad t > 0, \\ \partial_{\vec{n}} I = 0, & x \in \partial\Omega, \quad t > 0, \\ N = \int_\Omega (I + \frac{\gamma}{\beta} U), & t \geq 0. \end{cases} \tag{5.49}$$

By the comparison principle for ODEs and the first equation in (5.49), it follows that the sets Ω_-, Ω_0, and Ω_+ are all invariant for $U(t,\cdot)$. Also, the function $U(t,\cdot)$ is monotone increasing in t on Ω_- and has a nonnegative pointwise limit $U_-(\cdot)$ as $t \to +\infty$. Similarly, the function $U(t,\cdot)$ is monotone decreasing in t on Ω_+ and has a pointwise limit $U_+(\cdot)$ as $t \to +\infty$. On Ω_0, we have that $U(t,\cdot) = U(0,\cdot)$ for every $t > 0$. Define a function U^* on $\overline{\Omega}$ by

$$U^*(x) = \begin{cases} U_-(x), & \text{if} \quad x \in \Omega_-, \\ U(0,x), & \text{if} \quad x \in \Omega_0, \\ U_+(x), & \text{if} \quad x \in \Omega_+. \end{cases}$$

Then $U(t,x) \to U^*(x)$ as $t \to +\infty$ for every $x \in \overline{\Omega}$. Applying the Lebesgue Dominated Convergence Theorem, we have that

$$\lim_{t\to+\infty} \int_\Omega \frac{\gamma}{\beta} U(t,\cdot) = \int_\Omega \frac{\gamma}{\beta} U^*. \tag{5.50}$$

For each $0 < \eta < 1$, define the set

$$\Omega^*_\eta \coloneqq \{x \in \Omega \mid U^*(x) \leq 1 - \eta\}.$$

Notice that, by the continuity of measure,

$$\operatorname{meas}(\{x \in \Omega \mid U^*(x) < 1\}) = \lim_{\eta\to 0} \operatorname{meas}(\Omega^*_\eta). \tag{5.51}$$

Now, by (5.50), we have that

$$\begin{aligned}
\int_\Omega \frac{\gamma}{\beta} U^* &= \lim_{t\to\infty} \int_\Omega \frac{\gamma}{\beta} U(t,\cdot) \\
&= \lim_{t\to\infty} \int_\Omega S(t,\cdot) \\
&\leq N \\
&< \int_\Omega \frac{\gamma}{\beta}
\end{aligned}$$

which implies that $\operatorname{meas}(\{x \in \Omega \mid U^*(x) < 1\}) > 0$. Therefore, from (5.51), there exists some $0 < \eta_0 \ll 1$ such that $\operatorname{meas}(\Omega^*_{\eta_0}) > 0$. Now observe that $\Omega^*_{\eta_0} \subset \Omega_-$. Moreover, for each $x \in \Omega^*_{\eta_0}$, we have that $U(t,x) < U^*(x) \leq 1 - \eta_0$ for all $t > 0$. Hence,

$$\partial_t U = \beta(1-U)I \geq \eta_0 \beta I, \quad x \in \Omega^*_{\eta_0}, \quad t > 0.$$

Integrating in the time variable, we obtain

$$U(t,x) - U(0,x) \geq \eta_0 \beta \int_0^t I(s,x)\, ds.$$

Then, integrating over $\Omega^*_{\eta_0}$, we get

$$N \geq \int_{\Omega^*_{\eta_0}} (S(t,x) - S(0,x))\ dx = \int_{\Omega^*_{\eta_0}} \frac{\gamma}{\beta}(U(t,x) - U(0,x))\ dx \geq \sigma \int_0^t \int_{\Omega^*_{\eta_0}} I(s,x)\ dx\ ds$$

where $\sigma = \eta_0 \min_{x\in\overline{\Omega}} \gamma$. This yields that

$$\sup_{t>0} \int_0^t \int_{\Omega^*_{\eta_0}} I(s,x)\ dx\ ds \leq \frac{N}{\sigma}. \tag{5.52}$$

Note that $\|U(t,\cdot)\|_\infty \leq \max\{1, \|U(0,\cdot)\|_\infty\}$. Since $1 > 0$ and $U(t,x) > 0$ for all $t > 0$ and $x \in \overline{\Omega}$, then

$$|1 - U(t,x)| \leq \max\{1, U(t,x)\} \leq \max\{1, \|U(t,\cdot)\|_\infty\} \leq \max\{1, \|U(0,\cdot)\|_\infty\}$$

which implies that

$$\|\beta(1 - U(t,\cdot))\|_\infty \leq \|\beta\|_\infty \max\{1, \|U(0,\cdot)\|_\infty\}.$$

Furthermore, by Harnack's inequality for linear parabolic equations (see [16]*Theorem 2.5), there exists a positive constant C^* such that

$$I(t,x) \leq C^* I(t,y), \quad x,y \in \Omega, \quad t \geq 1$$

which gives us

$$\|I(t,\cdot)\|_\infty \leq \frac{C^*}{\text{meas}(\Omega^*_{\eta_0})} \int_{\Omega^*_{\eta_0}} I(t,x)\ dx, \quad t \geq 1. \tag{5.53}$$

After integrating (5.53) and combining with (5.52), we obtain

$$\int_1^\infty \|I(t,\cdot)\|_\infty\ dt \leq \frac{N}{\sigma} \frac{C^*}{\text{meas}(\Omega^*_{\eta_0})}. \tag{5.54}$$

However, using the regularity theory established for parabolic equations, we can see that the map $[1, +\infty) \ni t \mapsto I(t,\cdot) \in C^1(\Omega)$ is Hölder continuous. Hence, we are able to conclude from (5.54) that, as $t \to +\infty$, we have $\|I(t,\cdot)\|_\infty \to 0$. Observe that

$$\int_1^{+\infty} \|\partial_t U(t,\cdot)\|_\infty \ dt = \int_1^{+\infty} \|\beta(1-U(t,\cdot))I(t,\cdot)\|_\infty \ dt$$
$$\leq \frac{N}{\sigma}\|\beta\|_\infty \max\{1, \|U(0,\cdot)\|_\infty\} \frac{C^*}{\text{meas}(\Omega^*_{\eta_0})}.$$

Hence,

$$\int_1^{+\infty} \|\partial_t U(t,\cdot)\|_\infty \ dt < +\infty. \tag{5.55}$$

Now, since

$$U(t,x) = U(0,x) = \int_0^t \partial_s U(s,x)\ ds$$

then for $t_1 < t_2$, we have

$$\|U(t_2,x) - U(t_1,x)\|_\infty = \left\| \int_{t_1}^{t_2} \partial_s U(s,x)\ ds \right\|_\infty \leq \int_{t_1}^{t_2} \|\partial_s U(s,x)\|_\infty \ ds.$$

Observe that (5.55) implies that

$$\lim_{t_1,t_2 \to +\infty} \int_{t_1}^{t_2} \|\partial_s U(s,x)\|_\infty \ ds = 0.$$

Therefore,

$$\lim_{t_1,t_2 \to +\infty} \|U(t_2,x) - U(t_1,x)\|_\infty = 0.$$

Hence, $\{U(t,\cdot)\}_{t>0}$ is Cauchy in $C(\overline{\Omega})$ which means there exists $\tilde{U}^* \in C(\overline{\Omega})$ such that

$$\lim_{t\to+\infty} \left\| U(t,\cdot) - \tilde{U}^* \right\|_\infty = 0.$$

However, by construction $U(t,x) \to U^*(x)$ for all $x \in \overline{\Omega}$. By the uniqueness of limits, $U^* = \tilde{U}^*$ and so $U^* \in C\overline{\Omega}$. Furthermore, we have that $U(t,\cdot) \to U^*$ as $t \to +\infty$, uniformly on $\overline{\Omega}$. This also implies that $S(t,\cdot) \to S^* := \frac{\gamma}{\beta} U^*$ as $t \to +\infty$, uniformly on $\overline{\Omega}$, $S^* \in C(\overline{\Omega})$, and $\int_\Omega S^* = N$. Thus, we have shown (i).

(ii) Suppose we have an equilibrium solution (S, I) of (3.1). Then (S, I) satisfies

$$\begin{cases} (\gamma - \beta S)I = 0, & x \in \Omega, \\ d_I \Delta I = 0, & x \in \Omega, \\ \partial_{\vec{n}} I = 0, & x \in \partial\Omega, \\ N = \displaystyle\int_\Omega (S + I). \end{cases} \tag{5.56}$$

As such, I is identically equal to c for some $c \geq 0$. If $c > 0$ then, from the first equation in (5.56), we have that $S = \dfrac{\gamma}{\beta}$. Solving for I yields $I = N - \displaystyle\int_\Omega \frac{\gamma}{\beta}$. □

5.2 Proofs of Results from Chapter 4.

5.2.1 Proof of Theorem 4.1.1

Let $d_I > 0$ and define

$$\mathcal{R}_0^{\text{low}} = \inf_{l \geq l^*} \mathcal{N}_{d_I}(l),$$

where $\mathcal{N}_{d_I}(l)$ is defined by (2.18). It follows from (2.27) that $0 \leq \mathcal{R}_0^{\text{low}} \leq \{1, \overline{(\gamma/\beta)}\mathcal{R}_1\}$. Moreover, since $\mathcal{N}_{d_I}(l)$ for every $l \geq l^*$ and converges to a positive number as $l \to +\infty$, then $\mathcal{R}_0^{\text{low}} > 0$. We now prove (i) - (iii).

(i) If (S, I) is an endemic equilibrium solution of (1.5) for some $d_S > 0$, then by Lemma 2.3.1, $l := \dfrac{\kappa}{d_S} > l^*$, where κ is defined by (2.9). Furthermore, by Lemma 2.3.1, we have that

$$\mathcal{R}_0 = \mathcal{N}_{d_I}(\frac{\kappa}{d_S}) + \frac{\kappa}{l^*|\Omega|} \int_\Omega u^{\frac{\kappa}{d_S}} > \mathcal{R}_0^{\text{low}}.$$

Therefore, (1.5) has no endemic equilibrium solution whenever $\mathcal{R}_0 \leq \mathcal{R}_0^{\text{low}}$ and $d_S > 0$.

(ii) Now, suppose $\mathcal{R}_0 > \mathcal{R}_0^{\text{low}}$. Then, there is $l(\mathcal{R}_0, d_I) > l^*$ such that $\mathcal{N}_{d_I}(l(\mathcal{R}_0, d_I)) < \mathcal{R}_0$. Set

$$d_1^* := \frac{(\mathcal{R}_0 - \mathcal{N}_{d_I}(l(\mathcal{R}_0, d_I)))|\Omega| l^*}{l(\mathcal{R}_0, d_I) \int_\Omega u^{l(\mathcal{R}_0, d_I)}} > 0. \tag{5.57}$$

For each $d_S > 0$, consider the function $\mathcal{N}_{d_I, d_S}$ as defined in (5.7). Then, $\mathcal{N}_{d_I, d_S}$ is continuously differentiable in $l \geq l^*$. Moreover,

$$\mathcal{N}_{d_I,d_S}(l^*) = 1 \quad \text{and} \quad \lim_{l\to+\infty} N_{d_I,d_S}(l) = +\infty.$$

Now fix $0 < d_S < d_1^*$. Then, from (5.57), it follows that

$$\begin{aligned} N_{d_I,d_S}(l(\mathcal{R}_0, d_I)) &= \mathcal{N}_{d_I}(l(\mathcal{R}_0, d_I)) + \frac{d_S l(\mathcal{R}_0, d_I)}{l^*|\Omega|}\int_\Omega u^{l(\mathcal{R}_0,d_I)} \\ &< \mathcal{N}_{d_I}(l(\mathcal{R}_0, d_I)) + \frac{d_1^* l(\mathcal{R}_0, d_I)}{l^*|\Omega|}\int_\Omega u^{l(\mathcal{R}_0,d_I)} \\ &= \mathcal{R}_0. \end{aligned} \tag{5.58}$$

By the Intermediate Value Theorem, there is $l(\mathcal{R}_0, d_I, d_S) > l(\mathcal{R}_0, d_I)$ such that $\mathcal{N}_{d_I,d_S}(l(\mathcal{R}_0, d_I, d_S)) = \mathcal{R}_0$. Along with (5.8), this implies that the quantity

$$l_{\text{high}}(d_S) := \max\{l > l(\mathcal{R}_0, d_I) \mid \mathcal{N}_{d_I,d_S}(l) = \mathcal{R}_0\} \tag{5.59}$$

is a positive real number. Observe that

$$\mathcal{N}_{d_I,d_S}(l_{\text{high}}(d_S)) = \mathcal{R}_0 \quad \text{and} \quad \mathcal{N}_{d_I,d_S}(l) > \mathcal{R}_0, \quad l > l_{\text{high}}(d_S). \tag{5.60}$$

By Lemma 2.3.1,

$$(S_{\text{high}}, I_{\text{high}}) := (l_{\text{high}}(d_S)(1 - d_S u^{l_{\text{high}}(d_S)}), d_S l_{\text{high}}(d_S) u^{l_{\text{high}}(d_S)}) \tag{5.61}$$

is an endemic equilibrium solution of (1.5).

Finally, we show that (4.2) holds. Suppose that (S, I) is another endemic equilibrium solution of (1.5). By Lemma 2.3.1, we have that $\frac{\kappa}{d_S} > l^*$ and $I = d_S(\frac{\kappa}{d_S})u^{\frac{\kappa}{d_S}}$. Since the mapping $(l^*, +\infty) \ni l \mapsto lu^l$ is strictly increasing and $\mathcal{N}_{d_I,d_S}(\frac{\kappa}{d_S}) = \mathcal{R}_0 = \mathcal{N}_{d_I,d_S}(l_{\text{high}}(d_S))$, then $\frac{\kappa}{d_S} < l_{\text{high}}(d_S)$, which yields $I = d_S(\frac{\kappa}{d_S})u^{\frac{\kappa}{d_S}} < d_S l_{\text{high}}(d_S) u^{l_{\text{high}}(d_S)} = I_{\text{high}}$.

(iii) Suppose that $\mathcal{R}_0^{\text{low}} < 1$ and $\mathcal{R}_0^{\text{low}} < \mathcal{R}_0 < 1$. Let d_1^* be given by (5.57) and $l(\mathcal{R}_0, d_I)$ be as in the proof of (ii). Fix $0 < d_S < d_1^*$ and observe that $\mathcal{N}_{d_I,d_S}(l(\mathcal{R}_0, d_I)) < \mathcal{R}_0 < 1 = \mathcal{N}_{d_I,d_S}(l^*)$.

Therefore, by the Intermediate Value Theorem, there is $\tilde{l}(\mathcal{R}_0, d_I, d_S) \in (l^*, l(\mathcal{R}_0, d_I))$ such that $\mathcal{N}_{d_I,d_S}(\tilde{l}(\mathcal{R}_0, d_I, d_S)) = \mathcal{R}_0$. This implies that the quantity

$$l_{\text{low}}(d_S) := \min\{l \in [l^*, l(\mathcal{R}_0, d_I)) \mid \mathcal{N}_{d_I,d_S}(l) = \mathcal{R}_0\} \tag{5.62}$$

is well-defined and satisfies $l^* < l_{low}(d_S) < l(\mathcal{R}_0, d_I)$. Observe now that

$$\mathcal{N}_{d_I,d_S}(l_{low}(d_S)) = \mathcal{R}_0 \quad \text{and} \quad \mathcal{N}_{d_I,d_S}(l) > \mathcal{R}_0, \quad l^* \leq l < l_{low}(d_S). \tag{5.63}$$

By Lemma 2.3.1,

$$(S_{\text{low}}, I_{\text{low}}) := (l_{low}(d_S)(1 - d_S u^{l_{low}(d_S)}), d_S l_{low}(d_S) u^{l_{low}(d_S)}) \tag{5.64}$$

is an endemic equilibrium solution of (1.5). Since $l_{low}(d_S) < l(\mathcal{R}_0, d_I) < l_{high}(d_S)$ and the mapping $(l^*, +\infty) \ni l \mapsto lu^l$ is strictly increasing, then (4.3) holds. Moreover, it can be shown that any other endemic equilibrium solution of (1.5), if one exists, must satisfy (4.4). □

5.2.2 Proof of Proposition 4.1.2

It follows from Lemma 2.2.1 that $1 > \overline{(\gamma/\beta)}\mathcal{R}_1$ for every $d_I > \mathcal{R}_1^{-1}(1/\overline{(\gamma/\beta)})$. Hence, $1 > \overline{(\gamma/\beta)}\mathcal{R}_1 \geq \mathcal{R}_0^{\text{low}}$ for every $d_I > \mathcal{R}_1^{-1}(1/\overline{(\gamma/\beta)})$. □

5.2.3 Proof of Theorem 4.1.3

Proof of Theorem 4.1.3. Fix $d_I > 0$ and define

$$M_{d_I}^* := \frac{d_I}{|\Omega|} \sup_{l > l^*} \int_\Omega (u^l + lv^l),$$

where u^l and v^l are the unique positive solutions of (2.17) and (2.24), respectively, for each $l > l^*$. Note, from (2.21) and (2.26) that $u^l \to \dfrac{1}{d_I}$ and $lv^l \to 0$ as $l \to +\infty$, uniformly in Ω. Hence, $\int_\Omega (u^l + lv^l) \to \dfrac{|\Omega|}{d_I}$ as $l \to +\infty$. Note also, from (2.21) and (2.25) that $\int_\Omega (u^l + lv^l) \to \dfrac{(\int_\Omega \varphi_1)(\int_\Omega \beta\varphi_1^2)}{d_I \int_\Omega \beta\varphi_1^3}$ as

$l \to l^*$. Therefore, $\max\{1, \frac{(\int_\Omega \varphi_1)(\int_\Omega \beta\varphi_1^2)}{d_I \int_\Omega \beta\varphi_1^3}\} \le M^*_{d_I} < +\infty$. Defining

$$m^*_{d_I} \coloneqq \frac{1}{M^*_{d_I}},$$

then we have that $0 < m^*_{d_I} \le 1$. Hence, $d_{\text{low}} \coloneqq (1 - m^*_{d_I})d_I$ satisfies $0 \le d_{\text{low}} < d_I$.

Now, for each $d_S > 0$, consider the function $\mathcal{N}_{d_I,d_S}$ as defined in (5.7). Taking the derivative of the function $\mathcal{N}_{d_I,d_S}$ with respect to l, we get

$$\frac{d\mathcal{N}_{d_I,d_S}(l)}{dl} = \frac{1}{l^*|\Omega|}\left(|\Omega| + (d_S - d_I)\int_\Omega (lv^l + u^l)\right), \quad l > l^*. \tag{5.65}$$

From this point, we assume that $d_S > d_{\text{low}} = d_I(1 - m^*_{d_I})$. We will show that

$$\frac{d\mathcal{N}_{d_I,d_S}(l)}{dl} > 0, \quad l > l^* \tag{5.66}$$

If $d_S \ge d_I$, then it is clear that (5.66) follows from (5.65). So, suppose that $d_{\text{low}} < d_S < d_I$. Then, by (5.65), we have that

$$\begin{aligned}
\frac{d\mathcal{N}_{d_I,d_S}(l)}{dl} &= \frac{1}{l^*|\Omega|}\left(|\Omega| + (d_S - d_I)\int_\Omega (lv^l + u^l)\right) \\
&= \frac{1}{l^*}\left(1 - \left(1 - \frac{d_S}{d_I}\right)\frac{d_I}{|\Omega|}\int_\Omega (lv^l + u^l)\right) \\
&\ge \frac{M^*_{d_I}}{d_I l^*}(d_S - d_{\text{low}}) > 0
\end{aligned}$$

so that (5.66) holds when $d_{\text{low}} < d_S < d_I$ as well. Hence, the map $l \mapsto N_{d_I,d_S}(l)$ is strictly increasing on $[l^*, +\infty)$ when $d_S > d_{\text{low}}$. Thanks to Lemma 2.3.1, (1.5) has an endemic equilibrium solution if and only if $\mathcal{R}_0 > 1$. Moreover, in this case, when an endemic equilibrium solution exists, it is unique. □

5.2.4 Proof of Remark 4.1.1

Proof of Remark 4.1.1. Fix $d_I > 0$ and let $M^*_{d_I}$ and $m^*_{d_I}$ be as in the proof of Theorem 4.1.3 so that $d_{\text{low}} = d_I(1 - m^*_{d_I})$. Suppose that $d_{\text{low}} > 0$ and fix $0 < d_S < d_{\text{low}}$. Hence, $m^*_{d_I} < 1 - \frac{d_S}{d_I}$ which means

$0 < \dfrac{1}{1-\frac{d_S}{d_I}} < \dfrac{1}{m^*_{d_I}} = M^*_{d_I}$. Therefore, there is $l_0 > l^*$ such that $0 < \dfrac{1}{1-\frac{d_S}{d_I}} < \dfrac{d_I}{|\Omega|}\displaystyle\int_\Omega (u^{l_0} + l_0 v^{l_0})$, which implies that

$$\frac{d\mathcal{N}_{d_I,d_S}(l_0)}{dl} = \frac{1}{l^*}\left(1 - \left(1 - \frac{d_S}{d_I}\right)\frac{d_I}{|\Omega|}\int_\Omega (u^{l_0} + l_0 v^{l_0})\right) < 0. \tag{5.67}$$

On the other hand, we know from Lemma 2.4.2-(i)-(ii) that

$$\lim_{l\to+\infty} \frac{d\mathcal{N}_{d_I,d_S}(l)}{dl} = \frac{1}{l^*|\Omega|}\Big(|\Omega| + (d_S - d_I)\frac{|\Omega|}{d_I}\Big) = \frac{d_S}{d_I l^*} > 0. \tag{5.68}$$

Thanks to (5.67) and (5.68), we deduce that there are $l_0 < l_1 < l_2$ such that $\mathcal{N}_{d_I,d_S}(l_1) = \mathcal{N}_{d_I,d_S}(l_2)$. As a result, for $\mathcal{R}_0 = \mathcal{N}_{d_I,d_S}(l_1) = \mathcal{N}_{d_I,d_S}(l_2)$, we have from Lemma 2.3.1 that

$$(S_1, I_1) = (l_1(1 - d_I u^{l_1}), d_S l_1 u^{l_1}) \quad \text{and} \quad (S_2, I_2) = (l_2(1 - d_I u^{l_2}), d_S l_2 u^{l_2})$$

are two distinct endemic equilibrium solutions of (1.5). This completes the proof of the remark.

□

5.2.5 Proof of Theorem 4.2.1

Proof of Theorem 4.2.1. Suppose that (4.6) holds. By Lemma 2.4.2-(i)-(ii), we obtain

$$\frac{d\mathcal{N}_{d_I}(l^*)}{dl} = \Big(1 - (\overline{\varphi_1})\,(\overline{\beta\varphi_1^2})/\overline{\beta\varphi_1^3}\Big)/l^* < 0. \tag{5.69}$$

Thanks to (5.69), we can choose $d_2^* > 0$ small enough such that $\dfrac{d\mathcal{N}_{d_I,d_S}(l^*)}{dl} < 0$ for all $0 \le d_S < d_2^*$. Now, fix $0 < d_S < d_2^*$. Define the curve $\mathcal{F}_{d_I,d_S} : [l^*, +\infty) \to \mathbb{R}_+ \times [C(\overline{\Omega})]^2$ by

$$\mathcal{F}_{d_I,d_S}(l) = \Big(\mathcal{N}_{d_I,d_S}(l), l(1 - d_I u^l), d_S l u^l\Big), \quad l \ge l^*, \tag{5.70}$$

where u^l is the unique nonnegative stable solution of (2.17). Recalling that $\mathcal{R}_0 = N/(|\Omega| l^*)$, then $(\dfrac{N}{|\Omega|}, 0) = (l^*\mathcal{R}_0, 0)$ is the unique disease-free equilibrium solution of (1.5). Hence $(l^*, 0)$ is the unique disease-free equilibrium solution of (1.5) when $\mathcal{R}_0 = 1$. Observe also that $\mathcal{F}_{d_I,d_S}(l^*) =$

$(1, l^*, 0)$. By Lemma 2.3.1, system (1.5) has an endemic equilibrium solution (S, I) for some $\mathcal{R}_0 > 0$ if and only if $\mathcal{R}_0 = \mathcal{N}_{d_I,d_S}(l)$ and $(S, I) = (l(1 - d_I u^l), l d_S u^l)$ for some $l > l^*$. Therefore, as $\mathcal{R}_0$ increases from zero to infinity, the endemic equilibrium solutions of (1.5) are parametrized by the curve $\mathcal{F}_{d_I,d_S}$. This curve is simple and unbounded since the mapping $l \mapsto l u^l$ is strictly increasing with $\|l u^l\|_\infty \to +\infty$ as $l \to +\infty$. Furthermore, since $\frac{d\mathcal{N}_{d_I,d_S}(l^*)}{dl} < 0$, then the curve parametrized by $\mathcal{F}_{d_I,d_S}$ bifurcates from the left at $\mathcal{R}_0 = 1$.

□

5.2.6 Proof of Theorem 4.2.2

Note that the expression on the right-hand-side of (2.17) is analytic in the variables l and u. Hence, we can use the Implicit Function Theorem and the linear stability of u^l to derive that u^l is analytic in $l > l^*$. Hence, thanks to Lemma 2.4.2 and the limit (5.68), we have the following:

Lemma 5.2.1. *Fix $d_I > 0$ and $d_S > 0$. Consider the mapping $\mathcal{N}_{d_I,d_S}$ defined by (5.7) on $[l^*, +\infty)$. Then $\mathcal{N}_{d_I,d_S}$ is continuously differentiable on $[l^*, +\infty)$ and analytic on $(l^*, +\infty)$. Furthermore, if $\frac{d\mathcal{N}_{d_I,d_S}(l^*)}{dl} \neq 0$, then there exist m numbers $l_1^* = l^* < l_2^* < \cdots < l_m^* < l_{m+1}^* = +\infty$, $m \geq 1$, such that $\mathcal{N}_{d_I,d_S}$ is strictly monotone on $[l_i^*, l_{i+1}^*)$ for each $i = 1, \cdots, m$; $\mathcal{N}_{d_I,d_S}$ is strictly increasing on $[l_m^*, +\infty)$; and if $m \geq 2$, $\mathcal{N}_{d_I,d_S}$ changes its monotonicity at each l_i^*, $i = 2, \cdots, m$.*

Next, we give a proof of Theorem 4.2.2.

Proof of Theorem 4.2.2. Suppose that (4.7) holds. Then, by (2.27),

$$\lim_{l \to +\infty} \mathcal{N}_{d_I}(l) = \overline{(\gamma/\beta)}\mathcal{R}_1 < 1 = \mathcal{N}_{d_I}(l^*). \tag{5.71}$$

As a result, there exist some $\tilde{l}_0^* \gg l^*$ such that $\tilde{M}_{d_I}^* := \sup_{l \geq \tilde{l}_0^*} \mathcal{N}_{d_I}(l) < \mathcal{N}_{d_I}(l^*)$. We first set $\tilde{d}_3^* := |\Omega| l^* (\mathcal{N}_{d_I}(l^*) - \tilde{M}_{d_I}^*)/(\tilde{l}_0^* \int_\Omega u^{\tilde{l}_0^*})$. Next, note from (4.5), (2.25), and (2.26) that

$$\frac{d\mathcal{N}_{d_I}(l^*)}{dl} = \Big(1 - (\overline{\varphi_1})\big(\overline{\beta\varphi_1^2}\big)/\overline{\beta\varphi_1^3}\Big)\mathcal{R}_1 > 0. \tag{5.72}$$

Hence, there is $0 < d_3^* \leq \tilde{d}_3^*$ such that $\frac{d\mathcal{N}_{d_I,d_S}(l^*)}{dl} > 0$ for all $0 \leq d_S \leq d_3^*$. Now fix $0 < d_S < d_3^*$.

Let $m \geq 1$ be given by Lemma 5.2.1. Hence $\mathcal{N}_{d_I,d_I}$ is strictly increasing on $[l^*, l_2^*)$ and on $[l_m^*, +\infty)$. Observing that $\mathcal{N}_{d_I,d_S}(l^*) = \mathcal{N}_{d_I}(l^*) = 1$ and

$$\mathcal{N}_{d_I,d_S}(\tilde{l}_0^*) = \mathcal{N}_{d_I}(\tilde{l}_0^*) + \frac{\tilde{l}_0^* d_S}{l^*|\Omega|}\int_\Omega u^{\tilde{l}_0^*} < \mathcal{N}_{d_I}(\tilde{l}_0^*) + \frac{\tilde{l}_0^* \tilde{d}_3^*}{|\Omega| l^*}\int_\Omega u^{\tilde{l}_0^*} = \mathcal{N}_{d_I}(\tilde{l}_0^*) + \mathcal{N}_{d_I,d_S}(l^*) - \tilde{M}_{d_I}^* \leq \mathcal{N}_{d_I,d_S}(l^*),$$

then we must have that $m \geq 3$. Note that the simple connected curve $\mathcal{C}$ parametrized by (5.70) as in the proof of Theorem 4.2.1 consists of the endemic equilibrium solutions of (1.5). This time around, since $\mathcal{N}_{d_I,d_S}$ is increasing on $[l^*, l_2^*]$, then $\mathcal{C}$ bifurcates from the right at $\mathcal{R}_0 = 1$.

Next, since $\mathcal{N}_{d_I,d_S}$ is strictly increasing on $[l^*, l_2^*]$ and $[l_m^*, +\infty)$ and $\mathcal{N}_{d_I,d_S}(\tilde{l}_0^*) < \mathcal{N}_{d_I,d_S}(l_1^*)$, then $\mathcal{R}_{0,1}^{d_S} := \min\{\mathcal{N}_{d_I,d_S}(l_i^*) \mid i = 2, \cdots, 3\}$ is the global minimum value of $\mathcal{N}_{d_I,d_S}$ and is achieved at some $l_{i_0}^*$, $i_0 = 3, \cdots, m$. So, by Lemma 2.3.1, system (1.5) has no endemic equilibrium solution for $\mathcal{R}_0 < \mathcal{R}_{0,1}^{d_S}$ and $(l_{i_0}^*(1 - d_I u^{l_{i_0}^*}), d_S l_{i_0}^* u^{l_{i_0}^*})$ is an endemic equilibrium solution of (1.5) for $\mathcal{R}_0 = \mathcal{R}_{0,1}^{d_S}$. Thus, (i) and (ii) are proved. Next, set $\mathcal{R}_{0,2}^{d_S} = \mathcal{N}_{d_I,d_S}(l_2^*)$, and $\mathcal{R}_{0,3}^{d_S} = \max\{\mathcal{N}_{d_I,d_S}(l_i^*) \mid i = 1, \cdots, m\}$

(iii) First, suppose that $\mathcal{R}_0 \in (\mathcal{R}_{0,1}^{d_S}, 1]$. By the Intermediate Value Theorem, it follows as in (5.59) and (5.62) that both $l_{\text{high}}(d_S) > l_{i_0}^*$ and $l_{\text{low}}(d_S) \in (l_2^*, l_{i_0}^*)$ are well-defined. Moreover, $(S_{\text{high}}, I_{\text{high}})$ and $(S_{\text{low}}, I_{\text{low}})$ defined as in (5.61) and (5.64), respectively, are two distinct endemic equilibrium solutions of (1.5).

Next, suppose that $\mathcal{R}_0 = \mathcal{R}_{0,2}^{d_S}$. By the Intermediate Value Theorem, since

$$\mathcal{N}_{d_I,d_S}(l_{i_0}^*) < \mathcal{N}_{d_I,d_S}(l_2^*) = \mathcal{R}_{0,2}^{d_S} < \mathcal{R}_0$$

and $\mathcal{N}_{d_I,d_S}(l) \to +\infty$ as $l \to +\infty$, there is $l_{\text{high}}(d_S) > l_{i_0}^*$ such that $\mathcal{N}_{d_I,d_S}(l_{\text{high}}) = \mathcal{R}_0$. Hence, $(l_{\text{high}}(1 - d_I u^{l_{\text{high}}}), d_S l_{\text{high}} u^{l_{\text{high}}})$ and $(l_2^*(1 - d_I u^{l_2^*}), d_S l_2^* u^{l_2^*})$ are two distinct endemic equilibrium solutions of (1.5). This completes the proof of (iii)

(iv) Suppose that $1 < \mathcal{R}_0 < \mathcal{R}_{0,2}^{d_S}$. Observe that

$$\mathcal{N}_{d_I,d_S}(l^*) = 1 < \mathcal{R}_0 < \mathcal{R}_{0,2}^{d_S} = \mathcal{N}_{d_I,d_S}(l_2^*) \quad \text{and} \quad \mathcal{N}_{d_I,d_S}(l_{i_0}) < \mathcal{N}_{d_I,d_S}(l^*) < \mathcal{R}_0 < \mathcal{N}_{d_I,d_S}(l_2^*).$$

Hence, since $\mathcal{N}_{d_I,d_S}(l) \to +\infty$ as $l \to +\infty$, we can use the Intermediate Value Theorem to deduce the existence of minimal numbers $l_{\text{low},1} \in (l^*, l_2^*)$, $l_{\text{low},2} \in (l_2^*, l_{i_0}^*)$, and a maximal number $l_{\text{high}} > l_{i_0}^*$, such that $(S_{\text{low}}^i, I_{\text{low}}^i) := (l_{\text{low,i}}(1 - d_I u^{l_{\text{low,i}}}), d_S l_{\text{low,i}} u^{l_{\text{low,i}}})$, $i = 1, 2$, and $(S_{\text{high}}, I_{\text{high}}) := (l_{\text{high}}(1 - d_I u^{l_{\text{high}}}), d_S l_{\text{high}} u^{l_{\text{high}}})$ are three different endemic equilibrium solutions of (1.5). Clearly, $(S_{\text{low}}^1, I_{\text{low}}^1)$ and $(S_{\text{high}}, I_{\text{high}})$ are the minimal and maximal endemic equilibrium solutions of (1.5) in the sense of (4.2) and (4.4), respectively.

(v) If $\mathcal{R}_0 > \mathcal{R}_{0,2}^{d_S}$ then, since $\mathcal{N}_{d_I,d_S}(l) \to +\infty$ as $l \to +\infty$ then, as a consequence of the Intermediate Value Theorem, we have that there exists a $l_{\text{high}} > l_2^*$, such that $(l_{\text{high}}(1 - d_I u^{l_{\text{high}}}), d_S l_{\text{high}} u^{l_{\text{high}}})$ is an endemic equilibrium solution of (1.5). Now, suppose that $\mathcal{R}_0 > \mathcal{R}_{0,3}^{d_S}$. Then, since $\mathcal{N}_{d_I,d_S}$ is strictly increasing on $[l_m^*, +\infty)$, $\mathcal{N}_{d_I,d_S}(l_m^*) < \mathcal{R}_0$, and $\mathcal{N}_{d_I,d_S}(l) \to +\infty$ as $l \to +\infty$, there is a unique $\hat{l} > l_m^*$ such that $\mathcal{N}_{d_I,d_S}(\hat{l}) = \mathcal{R}_0$ and $\mathcal{N}_{d_I,d_S}(l) < \mathcal{R}_0$ for all $l \in [l_m^*, \hat{l})$. Moreover, observe that

$$\mathcal{N}_{d_I,d_S}(l) \le \mathcal{R}_{0,3}^{d_S} < \mathcal{R}_0, \quad l \in [l^*, l_m^*].$$

Therefore, $(\hat{l}(1 - d_I u^{\hat{l}}), d_S \hat{l} u^{\hat{l}})$ is the unique endemic equilibrium solution of (1.5).

Since $\mathcal{N}_{d_I,d_S}$ is strictly monotone increasing in d_S, we see that $\mathcal{R}_{0,i}^{d_S}$ is strictly increasing in d_S for each $i = 1, \cdots, 3$. Clearly, from the definition of $\mathcal{R}_{0,1}^{d_S} = \min_{l \ge l^*} \mathcal{N}_{d_I,d_S}(l)$, we get that $\mathcal{R}_{0,1}^{d_S} \to \inf_{l \ge l^*} \mathcal{N}_{d_I}(l) = \mathcal{R}_0^{\text{low}}$. Finally, from (5.72), we can find $\tilde{l}_0^{**} > l^*$ such that $\mathcal{N}_{d_I}$ is strictly increasing on $[l^*, \tilde{l}_0^{**}]$. As a result, we get that $\mathcal{N}_{d_I,d_S}(l) = \mathcal{N}_{d_I}(l) + (d_S l \int_\Omega u^l)/(|\Omega| l^*)$ is strictly increasing on $[l^*, \tilde{l}_0^{**}]$ for every $d_S > 0$ since u^l is strictly increasing in $l > l^*$. Therefore, for every $0 < d_S < d_3^*$, $l_2^* \ge \tilde{l}_0^{**}$ and $1 = \mathcal{N}_{d_I,d_S}(l^*) < \mathcal{N}_{d_I}(\tilde{l}_0^{**}) < \mathcal{N}_{d_I,d_S}(l_2^*) = \mathcal{R}_{0,2}^{d_S}$. As a result, $\mathcal{R}_{0,2}^* = \lim_{d_S \to 0} \mathcal{R}_{0,2}^{d_S} \ge \mathcal{N}_{d_I}(\tilde{l}_0^{**}) > 1$.

□

5.2.7 Proof of Theorem 4.3.1

Proof of Theorem 4.3.1. Fix $d_I > 0$ and suppose that $\mathcal{R}_0^{\text{low}} < 1$.

(i) Fix $\mathcal{R}_0^{\text{low}} < \mathcal{R}_0 < 1$ and let d_1^* be given by Theorem 4.1.1. For every $0 < d_S < d_1^*$, let $l_{\text{high}}(d_S)$ and $l_{\text{low}}(d_S)$ be defined by (5.59) and (5.62), respectively. First, we claim that

$$l_{\text{low}}(d_{S,1}) < l_{\text{low}}(d_{S,2}) \quad \text{for all} \quad 0 < d_{S,1} < d_{S,2} < d_1^*. \tag{5.73}$$

Indeed, fix $0 < d_{S,1} < d_{S,2} < d_1^*$. Then, since $\mathcal{N}_{d_I,d_{S,2}}(l_{\text{low}}(d_{S,2})) = \mathcal{R}_0$,

$$\mathcal{N}_{d_I,d_{S,1}}(l_{\text{low}}(d_{S,2})) = \mathcal{N}_{d_I,d_{S,2}}(l_{\text{low}}(d_{S,2})) - (d_{S,2} - d_{S,1})\frac{l_{\text{low}}(d_{S,2})}{l^*|\Omega|}\int_\Omega u^{l_{\text{low}}(d_{S,2})} < \mathcal{R}_0.$$

Hence, by (5.62) and (5.63), we have that (5.73) holds. Next, we claim that

$$l_{\text{high}}(d_{S,1}) > l_{\text{high}}(d_{S,2}) \quad \text{for all} \quad 0 < d_{S,1} < d_{S,2} < d_1^*. \tag{5.74}$$

Indeed, fix $0 < d_{S,1} < d_{S,2} < d_1^*$. Then, since $\mathcal{N}_{d_I,d_{S,2}}(l_{\text{high}}(d_{S,2})) = \mathcal{R}_0$

$$\mathcal{N}_{d_I,d_{S,1}}(l_{\text{high}}(d_{S,2})) = \mathcal{N}_{d_I,d_{S,2}}(l_{\text{high}}(d_{S,2})) - (d_{S,2} - d_{S_1})\frac{l_{\text{high}}(d_{S,2})}{|\Omega|l^*}\int_\Omega u^{l_{\text{high}}(d_{S,2})} < \mathcal{R}_0.$$

Hence, by (5.59) and (5.60), we have that (5.74) holds.

Due to (5.73) and (5.74), we have that

$$l_{\text{low}}^* = \lim_{d_S \to 0^+} l_{\text{low}}(d_S) = \inf_{0<d_S<d_1^*} l_{\text{low}}(d_S) < l(\mathcal{R}_0, N)$$

and

$$l_{\text{high}}^* := \lim_{d_S \to 0^+} l_{\text{high}}(d_S) = \sup_{0<d_S<d_1^*} l_{\text{high}}(d_S) > l(\mathcal{R}_0, d_I).$$

(i-1) Suppose that $\mathcal{R}_0/\mathcal{R}_1 < \overline{\gamma/\beta}$. We establish that (4.8) holds. First, we proceed by contradiction to show that

$$l_{\text{high}}^* < +\infty. \tag{5.75}$$

Indeed, if (5.75) were false, then $l_{\text{high}}(d_S) \to +\infty$ as $d_S \to 0$. As a result, it follows from Lemma 2.3.1-(i) that $\int_\Omega l_{\text{high}}(d_S)(1 - d_I u^{l_{\text{high}}(d_S)}) \to |\Omega|\overline{(\gamma/\beta)}$ as $d_S \to 0$. Moreover, since

$$\mathcal{R}_0 = \mathcal{N}_{d_I,d_S}(l_{\text{high}}(d_S)) > \frac{1}{|\Omega|l^*}\int_\Omega l_{\text{high}}(d_S)(1 - d_I u^{l_{\text{high}}(d_S)}) \quad \text{for all} \quad 0 < d_S < d_1^*,$$

we obtain that $\mathcal{R}_0 \geq \frac{1}{|\Omega|l^*}\lim_{d_S\to 0}\int_\Omega l_{\text{high}}(d_S)(1-d_I u^{l_{\text{high}}(d_S)}) = \overline{(\gamma/\beta)}\mathcal{R}_1$, which gives a contradiction.

Thus, (5.75) holds. This shows that there is a positive constant $C_1 = C_1(\mathcal{R}_0, d_I)$ such that

$$l_{\text{high}}(d_S) \le C_1, \quad 0 < d_S < \frac{d_1^*}{2}. \tag{5.76}$$

In particular,

$$I_{\text{low}} < I_{\text{high}} = d_S l_{\text{high}}(d_S) u^{l_{\text{high}}(d_S)} \le \frac{C_1}{d_I} d_S \quad \text{for all} \quad 0 < d_S < \frac{d_1^*}{2}. \tag{5.77}$$

Next, we claim that

$$l_{\text{low}}^* > l^*. \tag{5.78}$$

If (5.78) were false, then $l_{\text{low}}(d_S) \to l^*$ as $d_S \to 0$. This in turn implies that

$$\mathcal{R}_0 = \mathcal{N}_{d_I}(l_{\text{low}}(d_S)) + \frac{d_S l_{\text{low}}(d_S)}{|\Omega| l^*} \int_\Omega u^{l_{\text{low}}(d_S)} \to 1 \quad \text{as } d_S \to 0,$$

which contradicts our initial assumption that $\mathcal{R}_0 \neq 1$. Therefore, (5.78) holds. Thus, there is $C_2 = C_2(\mathcal{R}_0, d_I) > l^*$ such that

$$C_2 \le l_{\text{low}}(d_S) \le l(\mathcal{R}_0, d_I) \quad \text{for all} \quad 0 < d_S < \frac{d_1^*}{2}. \tag{5.79}$$

As a result, for all $0 < d_S < \frac{d_1^*}{2}$, we obtain that

$$I_{\text{low}} = d_S l_{\text{low}}(d_S) u^{l_{\text{low}}(d_S)} \ge C_2 d_S u^{C_2} \ge C_2 u_{\min}^{C_2} d_S \quad . \tag{5.80}$$

Combining (5.80) and (5.77) we derive that (4.8) holds.

Next, since $l_{\text{high}}^* > l_{\text{low}}^* > l^*$, then by Lemma 2.4.2-(i), we have that $S_{\text{low}} \to l_{\text{low}}^*(1 - d_I u^{l_{\text{low}}^*})$ and $S_{\text{high}} \to l_{\text{high}}^*(1 - d_I u^{l_{\text{high}}^*})$ as $d_S \to 0$ in $C^1(\overline{\Omega})$. On the other hand, since $N = \int_\Omega (S + I)$, we have that

$$N = l_{\text{low}}^* \int_\Omega (1 - d_I u^{l_{\text{low}}^*}) \quad \text{and} \quad N = l_{\text{high}}^* \int_\Omega (1 - d_I u^{l_{\text{high}}^*}).$$

Finally, since $l_{\text{low}}^* < l(\mathcal{R}_0, d_I) < l_{\text{high}}^*$, then $u_{\text{low}}^* := u^{l_{\text{low}}^*} < u^{l_{\text{high}}^*} =: u_{\text{high}}^*$.

(i-2) Suppose that $\mathcal{R}_0/\mathcal{R}_1 > \overline{\gamma/\beta}$. Note that the proof of (5.79) only uses the fact that

$\mathcal{R}_0 \neq 1$. Hence, $(S_{\text{low}}, I_{\text{low}})$ satisfies (4.8) and (4.10) as $d_S \to 0$. We claim that

$$l^*_{\text{high}} = +\infty. \tag{5.81}$$

Indeed, since $\mathcal{R}_0 > \overline{(\gamma/\beta)}\mathcal{R}_1 = \lim_{l \to +\infty} \mathcal{N}_{d_I}(l)$ then, for every $m > 1$, there is $l_m(\mathcal{R}_0, d_I) > m$ such that

$$\mathcal{N}_{d_I}(l_m(\mathcal{R}_0, d_I)) < \mathcal{R}_0.$$

Therefore, if we take $d_m(\mathcal{R}_0, d_I) := \dfrac{(\mathcal{R}_0 - \mathcal{N}_{d_I}(l_m(\mathcal{R}_0, d_I)))|\Omega| l^*}{l_m(\mathcal{R}_0, d_I) \int_\Omega u^{l_m(\mathcal{R}_0, d_I)}}$ for every $0 < d_S < d_m(\mathcal{R}_0, d_I)$, we can use the Intermediate Value Theorem and similar arguments as in (5.58) to conclude that there is $l_m(d_S) > l_m(\mathcal{R}_0, d_I)$ such that $\mathcal{N}_{d_I, d_S}(l_m(d_S)) = \mathcal{R}_0$. This shows that

$$l_{\text{high}}(d_S) \geq l_m(\mathcal{R}_0, d_S) > m, \quad 0 < d_S < d_m(\mathcal{R}_0, d_I).$$

Letting $m \to +\infty$ in the above inequality leads to (5.81). Thus, since $l(1 - d_I u^l) \to \dfrac{\gamma}{\beta}$ as $l \to +\infty$ in $C(\overline{\Omega})$ (see Lemma 2.4.2-(i)), we conclude that $S_{\text{high}} = l_{\text{high}}(d_S)(1 - d_I u^{l_{\text{high}}(d_S)}) \to \dfrac{\gamma}{\beta}$ as $d_S \to 0$ uniformly in $C(\overline{\Omega})$. Using the fact that $u^{l_{\text{high}}(d_S)} \to \dfrac{1}{d_I}$ as $d_S \to 0$ uniformly on Ω, we can observe that $d_S l_{\text{high}}(d_S) = \big(N - \int_\Omega S_{\text{high}}\big) / \int_\Omega u^{l_{\text{high}}(d_S)} \to d_I (N - \int_\Omega \frac{\gamma}{\beta})/|\Omega|$ as $d_S \to 0$ to conclude that $I_{\text{high}} \to \left(N - \int_\Omega \frac{\gamma}{\beta}\right)/|\Omega|$ as $d_S \to 0$ in $C(\overline{\Omega})$.

(ii) In addition, suppose that (4.7) holds. Let d_3^* and $\mathcal{R}^*_{0,2}$ be given by Theorem 4.2.2 and fix $1 < \mathcal{R}_0 < \mathcal{R}^*_{0,2}$. Hence, $1 < \mathcal{R}_0 < \mathcal{R}^{d_S}_{0,2}$ for every $0 < d_S < d_3^*$. Note from the proof of Theorem 4.2.2-(iv) that, for every $0 < d_S < d_3^*$, $(S^1_{\text{low}}, I^1_{\text{low}})$ and $(S_{\text{high}}, I_{\text{high}})$ are the minimal and maximal endemic equilibrium solutions of (1.5) in the sense of (4.4) and (4.2), respectively. Observing from (4.7) that $\mathcal{R}_0 > \overline{(\gamma/\beta)}\mathcal{R}_1$ then, by the similar argument as in (i-2), we have that $(S_{\text{high}}, I_{\text{high}})$ has the asymptotic profiles (4.12) as $d_S \to 0$. Next, observe that the constant number $\tilde{l}_0^*$ obtained in the proof of Theorem 4.2.2 depends only on d_I and satisfies $l_2^* < \tilde{l}_0^*$ for every $0 < d_S < d_3^*$. Therefore, $l_{\text{low},1} < \tilde{l}_0^*$ for every $0 < d_S < d_3^*$. Finally, since $\mathcal{R}_0 \neq 1$, we can proceed by the similar arguments leading to (5.78) to obtain that $C_2 := \liminf_{d_S \to 0} l_{\text{low},1} > 0$. In view of the preceding details, we see that $l_{\text{low},1}$ satisfies inequalities (5.79) with $l(\mathcal{R}_0, d_I)$ being replaced by $\tilde{l}_0^*$. So, $I_{\text{low},1}$ satisfies (4.8). Furthermore, up to a subsequence, S^1_{low} satisfies (4.10) as $d_S \to 0$. $\square$

CHAPTER 6

ONGOING WORKS, FUTURE WORKS, AND CONCLUSION

6.1 Conclusion

This dissertation focused on a diffusive SIS epidemic model where we have represented the action of infection using the mass-action transmission mechanism. In Chapter 3, we evaluated the effectiveness of control strategies centered on the restriction of population movement. In doing so, we discussed the well-posedness of our model as well as the existence of endemic equilibrium solutions. We examined the asymptotic profiles of these endemic equilibrium solutions and concluded that limiting only the movement of susceptible individuals can reduce the overall impact that the disease might have on the entire population. Hence, such a control strategy appears to be effective at reducing the spread of disease.

In Chapter 4, we looked at the question of possible non-uniqueness of endemic equilibrium solutions in the model. Unlike in the ODE model (1.1) or the model presented in [3], we found that the basic reproduction number cannot solely predict the dynamics of the infectious disease in the model (3.1). In particular, we showed that it is possible for the disease to persist even in the case that $\mathcal{R}_0 < 1$ which is a surprising result. Moreover, we found conditions on the parameters that lead to various bifurcation curves which suggest a multiplicity of endemic equilibrium solutions. We also provided a way to construct examples that satisfy the conditions presented in our results. Chapter 5 was dedicated entirely to the proofs of our main results.

6.2 Ongoing and Future Work

Current work has been on exploring the multiple-strain diffusive SIS model with mass-action. In the presence of multiple strains, new issues arise concerning competitive exclusion. In the case of competitive exclusion, it is possible for one strain of the disease to completely dominate the other strains. It might also be possible for two or more strains to coexistence and both persist within the population after enough time has passed. Our work on the multiple-strain model so far has been able to discover the precise conditions leading to both competitive exclusion and coexistence.

Future work is dedicated to expanding the diffusive SIS model into a diffusive SEIR model. In the SEIR model, we presume that susceptible people do not immediately become infected after being exposed to the disease. Once exposed to the disease, the susceptible person is moved to the exposed group where there is some chance that they do not become infected and, instead, return to the susceptible group. By including the recovered group, we make the additional assumption that people gain complete immunity after being first infected then recovering from the disease. So far, our preliminary results have been promising and demonstrate the striking differences between the diffusive SIS model and the diffusive SEIR model.

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